ECE2311D10 Exam 3

Your Name: SOLUTION Your box #: ____________

April 9, 2010

- Open book, open notes.
- Calculators permitted.
- Look over all of the questions before starting.
- Budget your time to allow yourself enough time to work on each question.
- Write neatly and show your work/reasoning!
- This exam is worth a total of 100 points.

problem 1	problem 2	problem 3	TOTAL
20 points	50 points	30 points	100 points

1. 20 points. For each of the following signals, determine which are periodic and which are aperiodic. For those that are periodic, determine T_0 and ω_0. For those that are not periodic, briefly explain why.

Signal	Periodic? (Y/N)	Fundamental period T_0	Fundamental frequency ω_0
$\cos(2\pi t)u(t)$	N	The $u(t)$ term causes this signal to not repeat	
$\cos(6t) + 2\sin(9t)$	Y	$T_0 = \frac{2\pi}{\omega_0} = \frac{2\pi}{3}$ $\cancel{T_0 = 3}$ ~~(greatest common factor)~~	$\omega_0 = 3$ (greatest common factor) $\cancel{\omega_0 = \frac{2\pi}{T_0} = \frac{2\pi}{3}}$ WHOOPS!
$\cos^2(t)$	Y	$T_0 = \frac{2\pi}{\omega_0} = \pi$	$\omega_0 = 2$
$\cos(2\pi t) + \sin(t)$	N	The ratio of the two frequencies present in the signal is not rational.	

$\cos^2 t = \frac{1}{2}(1 + \cos 2t)$

2. 50 points total. Suppose you have the periodic signal $x(t)$ given in Figure 1 below.

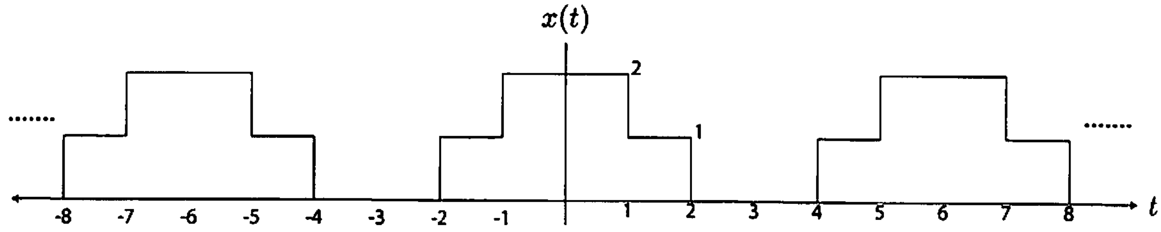

Figure 1: A periodic signal.

(a) 35 points. Write $x(t)$ as a trigonometric Fourier series and compute expressions for all of the coefficients. You should be able to perform the necessary integrals.

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)$$

$$\boxed{\begin{aligned} T_0 &= 6 \\ \omega_0 &= \frac{2\pi}{T_0} = \frac{2\pi}{6} = \frac{\pi}{3} \end{aligned}}$$

Recognize that signal is even symmetric $\Rightarrow$ all b_n terms will be equal to zero. $\boxed{b_n = 0}$

$$\boxed{a_0 : \frac{1}{6}\int_{-3}^{3} x(t)\,dt = \frac{1}{6}\cdot 6 = 1}$$

$$a_n = \frac{2}{6}\int_{-3}^{3} x(t)\cos(n\omega_0 t)\,dt \overset{\text{even}}{=} \frac{4}{6}\int_{0}^{3} x(t)\cos(n\omega_0 t)\,dt$$

$$= \frac{4}{6}\left\{\int_0^1 2\cos(n\omega_0 t)\,dt + \int_1^2 1\cos(n\omega_0 t)\,dt\right\}$$

$$= \frac{4}{6}\left\{\frac{2}{n\omega_0}\left(\sin(n\omega_0) - \sin(0)\right) + \frac{1}{n\omega_0}\left(\sin(2n\omega_0) - \sin(n\omega_0)\right)\right\}$$

$$= \frac{4}{6n\omega_0}\left[\sin(n\omega_0) + \sin(2n\omega_0)\right]$$

Recall $\omega_0 = \frac{\pi}{3} \Rightarrow a_n = \frac{4}{6n\frac{\pi}{3}}\left[\sin\left(\frac{n\pi}{3}\right) + \sin\left(\frac{2n\pi}{3}\right)\right]$

$$\boxed{a_n = \frac{2}{n\pi}\left[\sin\left(\frac{n\pi}{3}\right) + \sin\left(\frac{2n\pi}{3}\right)\right]}$$

(b) 15 points. Write $x(t)$ as a *compact* trigonometric Fourier series and compute expressions for all of the coefficients. Hint: You can compute these coefficients without integration.

$$C_n = \sqrt{a_n^2 + b_n^2} = |a_n|$$

$$\Rightarrow \boxed{C_n = \frac{2}{n\pi} \left| \sin\left(\frac{n\pi}{3}\right) + \sin\left(\frac{2n\pi}{3}\right) \right|}$$

$$\theta_n = \tan^{-1}\left(\frac{-b_n}{a_n}\right) = \begin{cases} 0 & a_n \geq 0 \\ \pi & a_n < 0 \end{cases}$$

since $b_n = 0 \ \forall n$

We can simplify this a bit by looking at the trend

n	$\sin(\frac{n\pi}{3}) + \sin(2n\frac{\pi}{3})$
1, 7, ..	1.7321
2, 8, ...	0
3, 9, ...	0
4, 10, ..	0
5, 11, ..	−1.7321
6, 12, ...	0

hence

$$C_n = \begin{cases} \frac{1.1027}{n} & n = 1, 5, 7, 11, 15, \ldots \\ 0 & \text{otherwise.} \end{cases}$$

$$\theta_n = \begin{cases} \pi & n = 5, 11, 17, \ldots \\ 0 & \text{otherwise.} \end{cases}$$

3. 30 points total. Suppose you have a periodic signal with trigonometric Fourier series representation

$$x(t) = c_0 + c_1 \cos(2\pi t + \theta_1) + c_2 \cos(4\pi t + \theta_2)$$

$\omega_0 = 2\pi$

$T_0 = 1$

with $c_0 = 1$ and

$$c_n = \begin{cases} \frac{1}{n^2} & n = 1, 2 \\ 0 & \text{otherwise} \end{cases}$$

and

$$\theta_n = \begin{cases} -n\frac{\pi}{2} & n = 1, 2 \\ 0 & \text{otherwise} \end{cases}$$

and you apply this signal as the input to the circuit shown in Figure 2 below.

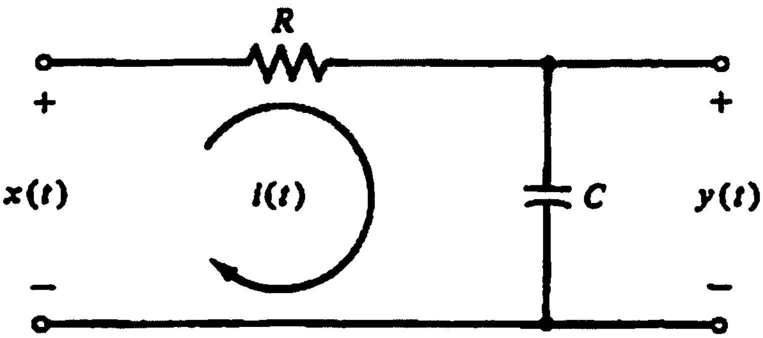

Figure 2: A circuit.

Suppose $R = C = 1$. Write an exact expression for $y(t)$.

$$H(\omega) = \frac{\frac{1}{j\omega C}}{R + \frac{1}{j\omega C}} = \frac{\frac{1}{j\omega}}{1 + \frac{1}{j\omega}} = \frac{1}{1 + j\omega} \Rightarrow H(n\omega_0) = \frac{1}{1 + jn\omega_0}$$

Note That $\omega_0 = 2\pi$

We need $|H(n\omega_0)|$ and $\measuredangle H(n\omega_0)$ for $n = 0, 1, 2$

n	$\lvert H(n\omega_0)\rvert$	$\measuredangle H(n\omega_0)$
0	1	—
1	0.1572	−1.4130 radians
2	0.0793	−1.4914 radians

Hence $y(t) = 1 + 0.1572 \cdot c_1 \cos(2\pi t + \theta_1 - 1.413)$
$+ 0.0793 \cdot c_2 \cos(4\pi t + \theta_2 - 1.4914)$

$$\boxed{y(t) = 1 + 0.1572 \cos(2\pi t - 2.9838) + 0.0198 \cos(4\pi t - 4.6330)}$$