ECE2311D10 Exam 4

Your Name: SOLUTION Your box #: 

April 16, 2010

- Open book, open notes.
- Calculators permitted.
- Look over all of the questions before starting.
- Budget your time to allow yourself enough time to work on each question.
- Write neatly and show your work/reasoning!
- This exam is worth a total of 100 points.
1. 40 points total. Suppose you have the aperiodic signal $x_1(t)$ given in Figure 1 below.

\[
x_1(t)
\]

![Figure 1: An aperiodic signal.](image)

(a) 30 points. Compute the Fourier transform of $x_1(t)$, i.e. compute $X_1(\omega) = \mathcal{F}\{x_1(t)\}$. Simplify your answer as much as possible using Euler's identity and other trigonometric identities. You may find this particular trigonometric identity useful:

\[
\begin{align*}
\sin^2(\nu) &= \frac{1 - \cos(2\nu)}{2} \\
X_1(\omega) &= \int_{-\infty}^{\infty} x_1(t)e^{-j\omega t}dt = \int_{-2}^{0} (-1)e^{-j\omega t}dt + \int_{0}^{2} (1)e^{-j\omega t}dt \\
&= \frac{-1}{-j\omega} e^{-j\omega t} \bigg|_{-2}^{0} + \frac{1}{-j\omega} e^{-j\omega t} \bigg|_{0}^{2} \\
&= \frac{1}{j\omega} \left[ 1 - e^{j2\omega} - \left( e^{-j2\omega} - 1 \right) \right] \\
&= \frac{1}{j\omega} \left[ 2 - \left( e^{j2\omega} + e^{-j2\omega} \right) \right] \\
&= \frac{2}{j\omega} \left[ 1 - \cos(2\omega) \right] = \frac{4}{j\omega} \sin^2(\omega) \\
\Rightarrow \quad X_1(\omega) &= \frac{4}{j\omega} \sin^2(\omega)
\end{align*}
\]
(b) 10 points. Carefully sketch the magnitude and phase spectra of the aperiodic signal $x_1(t)$.

\[ |X_1(\omega)| = \frac{4}{|\omega|} \sin^2(\omega) \]

Plotting this in Matlab will show that these points are not as sharp as I have drawn them, but they touch zero at the correct places.

\[ \text{Arg } X_1(\omega) = \begin{cases} -\frac{\pi}{2} & \omega > 0 \\ 0 & \omega = 0 \\ \frac{\pi}{2} & \omega < 0 \end{cases} \text{ (undefined really)} \]
2. 30 points. Suppose you have the periodic signal $x_2(t)$ given in Figure 2 below.

![Figure 2: A periodic signal.](image)

Using your result from Problem 1, write $x_1(t)$ as an exponential Fourier series and give an expression for the Fourier coefficients. You do not need to perform any integrals to determine the Fourier coefficients.

Here you just need to use the fact that

$$T_0 D_n = X_1(n\omega_0) \quad \text{where} \quad T_0 = 6 \quad \text{and} \quad \omega_0 = \frac{\pi}{3}$$

So

$$D_n = \frac{-4j}{6n\pi^2} \sin^2 \left(\frac{n\pi}{3}\right) = \frac{-2j}{n\pi^2} \sin^2 \left(\frac{n\pi}{3}\right)$$

and

$$x_2(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\pi t/3}$$

You can confirm these are the correct $D_n$ also by doing the integration directly, i.e.

$$D_n = \frac{1}{T_0} \int_{T_0} x_2(t) e^{-jn\omega_0} dt$$

but this would be repeating the work you did in problem 1.
3. 30 points. Suppose you apply the periodic signal \( x_2(t) \) as an input to a system with impulse response

\[
h(t) = e^{-t} u(t)
\]

(1)

where \( u(t) \) is the usual unit step function. Using your results from Problem 2, write the output of this system \( y(t) \) as an exponential Fourier series.

From the table on page 702, we have

\[
e^{-at} u(t) \leftrightarrow \frac{1}{a + j\omega}
\]

hence

\[
H(\omega) = \frac{1}{1 + j\omega} = \frac{1 - j\omega}{1 + \omega^2}
\]

and

\[
H(n\omega_0) = \frac{1 - jn\frac{\pi}{3}}{1 + (n\frac{\pi}{3})^2} = \frac{9 - jn3\pi}{9 + n^2\pi^2}
\]

The output of the system is then

\[
y(t) = \sum_{n=-\infty}^{\infty} D_n H(n\omega_0) e^{j n\omega_0 t}
\]

hence

\[
y(t) = \sum_{n=-\infty}^{\infty} \frac{(-2j)(9 - jn3\pi)}{(n\pi)(9 + n^2\pi^2)} \sin^2\left(n\frac{\pi}{3}\right) e^{j n\frac{\pi}{3} t}
\]

\[
y(t) = \sum_{n=-\infty}^{\infty} \frac{(6\pi n + 18j)}{9\pi n + n^3\pi^3} \sin^2\left(n\frac{\pi}{3}\right) e^{j n\frac{\pi}{3} t}
\]