ECE2311D10 Exam 5

Your Name: SOLUTION
Your box #: 

April 23, 2010

- Open book, open notes.
- Calculators permitted.
- Look over all of the questions before starting.
- Budget your time to allow yourself enough time to work on each question.
- Write neatly and show your work/reasoning!
- This exam is worth a total of 100 points.

<table>
<thead>
<tr>
<th>problem 1</th>
<th>problem 2</th>
<th>problem 3</th>
<th>TOTAL</th>
</tr>
</thead>
<tbody>
<tr>
<td>30 points</td>
<td>30 points</td>
<td>40 points</td>
<td>100 points</td>
</tr>
</tbody>
</table>
1. 30 points. Find the Fourier transform of the aperiodic signal \( x(t) \) shown in Figure 1 below. Note that
\[
\frac{d}{dt} x(t) = \frac{1}{2} \text{rect} \left( \frac{t}{4} \right) - 2\delta(t).
\]

You can compute the answer to this problem without integration by using the Fourier transform table and properties in your book. **Simplify your answer as much as possible.**

![Figure 1: An aperiodic signal.](image)

Let \( x_1(t) = \frac{1}{2} \text{rect} \left( \frac{t}{4} \right) - 2\delta(t) \)

then, from the table on p. 702,
\[
X_1(\omega) = \frac{1}{2} \cdot 4 \sin\left( \frac{\omega \cdot 4}{2} \right) - 2
\]
\[
X_1(\omega) = 2 \sin\left( 2\omega \right) - 2
\]
since \( x_1(t) = \frac{d}{dt} x(t) \)

\[
X_1(\omega) = j\omega X(\omega) \quad \text{(time differentiation property)}
\]

hence
\[
X(\omega) = \frac{2}{j\omega} \left[ \sin\left( 2\omega \right) - 1 \right] = \frac{2j}{\omega} \left[ 1 - \sin\left( 2\omega \right) \right]
\]

2
2. 30 points. Find the inverse Fourier transform of $X(\omega)$ shown in Figure 2 below. Simplify your answer as much as possible using Euler's identity and other trigonometric identities. 

Hint: You can save some time here by using one or more properties of the Fourier transform.

![Graph of $X(\omega)$ showing two rectangular pulses centered at $-3$ and $3$ with a magnitude of 1.](image)

**Figure 2:** Frequency domain representation. Note that $\angle X(\omega) = 0$ for all $\omega$.

Line 18 of table: $\frac{W}{\pi} \text{sinc}(Wt) \leftrightarrow \text{rect}\left(\frac{W}{2W}\right)$

\[
\begin{align*}
X_1(\omega) & \quad \text{this is } \text{rect}\left(\frac{W}{2}\right) \text{ hence } W=1 \\
\end{align*}
\]

Hence $x_1(t) = \frac{1}{\pi} \text{sinc}(t)$

Now use the frequency shifting property...

\[
\begin{align*}
x(t) &= \frac{1}{\pi} \text{sinc}(t) e^{j4t} + \frac{1}{\pi} \text{sinc}(t) e^{-j4t} \\
&= \frac{1}{\pi} \text{sinc}(t) \left[ e^{j4t} + e^{-j4t} \right] \\
&= \frac{1}{\pi} \text{sinc}(t) 2 \cos(4t)
\end{align*}
\]

\[
x(t) = \frac{2}{\pi} \cos(4t) \text{sinc}(t)
\]
3. 40 points total. Suppose you apply an input signal \( x(t) = e^{-3(t-1)}u(t-1) \) to a linear time-invariant system with impulse response \( h(t) = e^{-t}u(t) \). Compute the output of the system \( y(t) \) using Fourier analysis.

(a) Let \( v(t) = e^{-3t}u(t) \) then \( V(\omega) = \frac{1}{3+j\omega} \) (from line 1 of the table)

Since \( x(t) = v(t-1) \) we can use the time shifting property to write

\[
X(\omega) = V(\omega)e^{-j\omega} = \frac{e^{-j\omega}}{3+j\omega}
\]

\( H(\omega) \) can be also computed from line 1 of the table

\[
H(\omega) = \frac{1}{1+j\omega}
\]

Hence \( Y(\omega) = \frac{e^{-j\omega}}{(1+j\omega)(3+j\omega)} \)

(b) To compute \( y(t) \), we need to perform partial fraction expansion

\[
e^{-j\omega}\left[\frac{K_1}{1+j\omega} + \frac{K_2}{3+j\omega}\right] = e^{-j\omega}\left[\frac{1}{(1+j\omega)(3+j\omega)}\right]
\]

\[
K_1(3+j\omega) + K_2(1+j\omega) = 1 + 0j\omega
\]

\[
\begin{align*}
3K_1 + K_2 &= 1 \\
K_1 + K_2 &= 0
\end{align*}
\]

\( \Rightarrow \frac{2K_1}{1} = 1 \Rightarrow K_1 = \frac{1}{2} \) \( \Rightarrow K_2 = -\frac{1}{2} \)

Hence \( Y(\omega) = e^{-j\omega}\left[\frac{\frac{1}{2}}{1+j\omega} - \frac{\frac{1}{2}}{3+j\omega}\right] \Rightarrow \frac{1}{2}e^{-1(t-1)}u(t)

\( -\frac{1}{2} \cdot \frac{e^{-3(t-1)}}{u(t)} \) = \( \frac{1}{2}\left[e^{-(t-1)} - e^{-3(t-1)}\right]u(t-1) \)