1. Lathi 1.1-9

This $x(t)$ is not a periodic signal.

Let's compute the energy first...

$$E_x = \int_{-\infty}^{\infty} x^2(t)\,dt = \int_{0}^{\infty} x^2(t)\,dt$$ because $x(t)=0$ for all $t<0$

$$= \int_0^2 x^2(t)\,dt + \int_2^5 x^2(t)\,dt + \int_5^9 x^2(t)\,dt + \ldots$$

in each of these integration regions, $x(t)$ is on for the first second and off for the remaining time. Hence, each integral evaluates to be equal to one.

$$E_x = \sum_{n=0}^{\infty} \int_{t_n}^{t_n+1} x^2(t)\,dt = \sum_{n=0}^{\infty} 1 = \boxed{\infty = E_x}$$

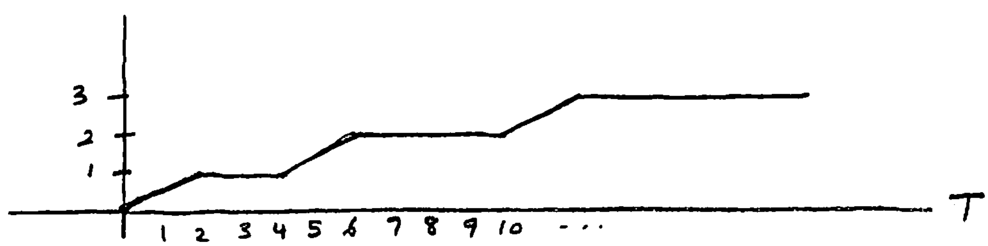
where $t_n = 0, 2, 5, 9, 14, \ldots$

Now power...

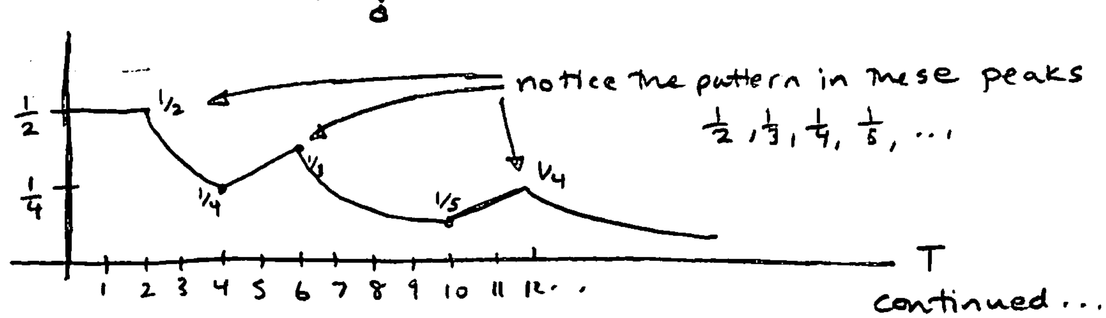
$$P_x = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x^2(t)\,dt = \lim_{T\to\infty} \frac{1}{T} \int_0^{T/2} x^2(t)\,dt$$

because $x(t)=0$ for all $t<0$

Lets look at $\int_0^{T/2} x^2(t)\,dt$



Now lets sketch $\frac{1}{T}\int_0^{T/2} x^2(t)\,dt$



continued...

problem 1 continued...

The trend is evident here, but we can formalize our result by noting that

$$\frac{1}{T}\int_0^{T/2} x^2(t)\,dt \le \frac{1}{n} \quad \text{for} \quad (n-1)! \le T \le n!$$

e.g. suppose $n=2$, we see that $\frac{1}{T}\int_0^{T/2} x^2(t)dt \le \frac{1}{2}$ for $1 \le T \le 2$ from the graph.

Hence $\lim_{T\to\infty} \frac{1}{T}\int_0^{T/2} x^2(t)dt$ must be zero.

$$\Rightarrow \boxed{P_x = 0.}$$

2. a) Use KVL to write

$$x(t) = Ri(t) + y(t)$$

We know that $i(t) = C\frac{dy(t)}{dt}$

$$\Rightarrow \boxed{RC\frac{dy(t)}{dt} + y(t) = x(t)}$$

in Lathi's notation $(RCD+1)\,y(t) = x(t)$

b) Following the approach on pp. 164-165, we have $Q(D) = RCD+1$ and $P(D) = 1$

We need to rewrite these polynomials in the correct form by dividing both sides by RC.

diff equation: $\left(D + \frac{1}{RC}\right) y(t) = \frac{1}{RC} x(t)$

$Q(D) = D + \frac{1}{RC}$ with $a_1 = \frac{1}{RC}$

$P(D) = 0\cdot D + \frac{1}{RC}$ with $b_0 = 0$ and $b_1 = \frac{1}{RC}$

Since $b_0 = 0$, $h(t)$ consists only of characteristic modes.

The single root of $Q(D)$ is $\lambda + \frac{1}{RC} = 0 \Rightarrow \lambda = -\frac{1}{RC}$

Hence $h(t) = b e^{-\frac{1}{RC}t} u(t)$. We need to find b...

To find K_1
we let $x(t) = \delta(t)$ and $y(t) = h(t)$ to write

$$\dot{h}(t) + \frac{1}{RC} h(t) = \frac{1}{RC}\delta(t)$$

initial condition: $h(0^-)$ is zero (relaxed initial cond.)

but $h(0^+) = K_1$. Hence $\dot{h}(0) = K_1 \delta(t)$.

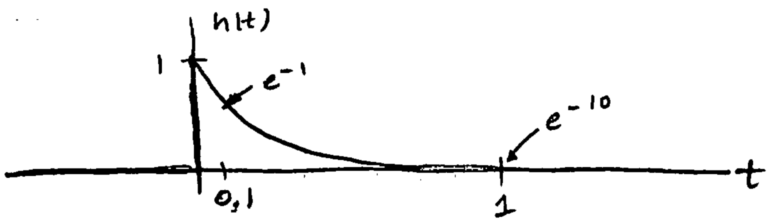
Match coefficients of impulse terms on both sides

$$K_1 \delta(t) + (\text{not an impulse}) = \frac{1}{RC}\delta(t)$$

Hence $K_1 = \frac{1}{RC}$.

Hence $h(t) = \frac{1}{RC} e^{-\frac{1}{RC}t} u(t)$ is the impulse response.

When $R=1$ and $C=0.1$, $h(t) = e^{-10t} u(t)$



c) Step response can be found by convolution

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, \underbrace{h(t-\tau)}_{\text{impulse response from part b}}\, d\tau \qquad x(\tau) = u(\tau)$$

The steps follow those in the 23-Mar lecture

$$y(t) = \int_0^{\infty} h(t-\tau)\,d\tau = \begin{cases} \int_0^t \frac{1}{RC} e^{-\frac{1}{RC}(t-\tau)}\,d\tau & t \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

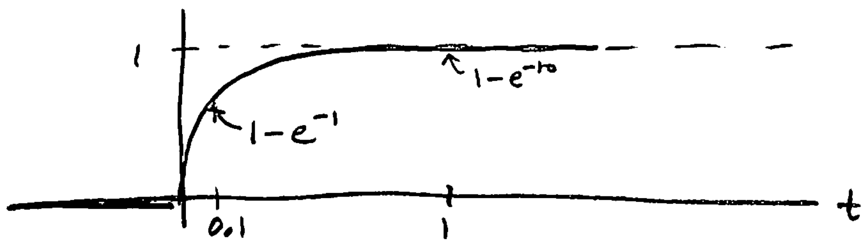
$$= \begin{cases} e^{-\frac{t}{RC}} \int_0^t \frac{1}{RC} e^{\frac{\tau}{RC}}\,d\tau & t \geq 0 \\ 0 & \text{otherwise} \end{cases} = \begin{cases} e^{-\frac{t}{RC}}\left[e^{\frac{t}{RC}} - 1\right] \\ 0 & \text{otherwise} \end{cases}$$

continued...

$$y(t) = \begin{cases} 1 - e^{-t/RC} & \\ 0 & \text{otherwise.} \end{cases} = \left[1 - e^{-t/RC}\right] u(t)$$

This is the step response

When $R=1$ and $C=0.1$, we have



3. Lathi 2.4-6

$$\int_{-\infty}^{\infty} u(\tau)\sin(t-\tau)u(t-\tau)\,d\tau = \sin t\, u(t) * u(t)$$

From $u(\tau)$ and $u(t-\tau)$ we have the limits of integration between 0 and t

$$= \begin{cases} \int_0^t \sin(t-\tau)\,d\tau & t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} 1 - \cos(t) & t \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

$$\cos t\, u(t) * u(t) = \int_{-\infty}^{\infty} u(\tau)\cos(t-\tau)u(t-\tau)\,d\tau$$

Same idea...

$$= \begin{cases} \int_0^t \cos(t-\tau)\,d\tau & t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} \sin(t) & t \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

5. Lathi 2.4-7 $\quad h(t) = e^{-t}u(t)$

a) $x(t) = u(t)$

$$y(t) = \left(1 - e^{-t}\right)u(t)$$

This was done in lecture, but you can do this with line 2 of the table setting $\lambda = -1$.

b) $x(t) = e^{-t}u(t)$ $\quad$ use line 5 of the table with $\lambda = -1$

$$y(t) = t\,e^{\lambda t}u(t) = te^{-t}u(t)$$

c) $x(t) = e^{-2t}u(t)$ $\quad$ use line 4 of the table with $\lambda_1 = -2$ and $\lambda_2 = -1$

$$y(t) = \frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2}\,u(t)$$

$$= \left(e^{-t} - e^{-2t}\right)u(t)$$

d) $x(t) = \sin(3t)\,u(t)$

Use euler's identity : $\sin x = \frac{1}{2j}\left[e^{jx} - e^{-jx}\right]$

We can use line 2 and linearity to compute the result.

$$x_1(t) = \frac{1}{2j}e^{j3t}u(t) \longrightarrow \boxed{S} \longrightarrow y_1(t) = \frac{1}{2j}\left[\frac{e^{j3t} - e^{-t}}{j3+1}\right]u(t)$$

$h(t) = e^{-t}u(t)$

$$x_2(t) = \frac{-1}{2j}e^{-j3t}u(t) \longrightarrow \boxed{S} \longrightarrow y_2(t) = \frac{-1}{2j}\left[\frac{e^{-j3t} - e^{-t}}{-j3+1}\right]u(t)$$

Note that we've used the homogeneity property here. Now lets use additivity

$$x(t) = \sin(3t)u(t) = x_1(t) + x_2(t)$$

$$y(t) = y_1(t) + y_2(t) = \frac{1}{2j}\left[\frac{e^{j3t} - e^{-t}}{j3+1} - \frac{e^{-j3t} - e^{-t}}{-j3+1}\right]u(t)$$

simplify...

Note that $(1+j3)(1-j3) = 1+9 = 10$

So $y(t) = \frac{1}{20j}\left[\left(e^{j3t} - e^{-t}\right)(1-j3) - \left(e^{-j3t} - e^{-t}\right)(1+j3)\right]u(t)$

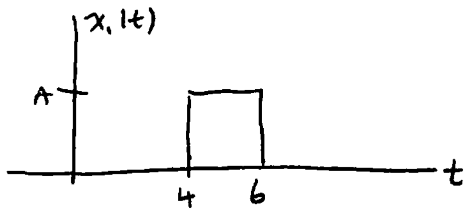
$$= \frac{1}{20j}\left[e^{j3t} \cancel{- e^{-t}} - e^{-j3t} \cancel{+ e^{-t}} + j3\left(-e^{j3t} + e^{-t} - e^{-j3t} + e^{-t}\right)\right]u(t)$$

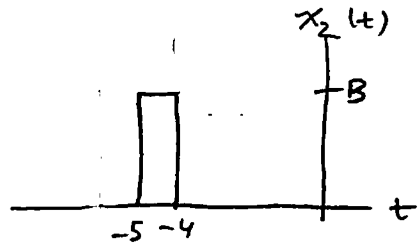
$$= \frac{1}{20j}\left[e^{j3t} - e^{-j3t}\right]u(t) + \frac{3}{20}\left[2e^{-t} - e^{-j3t} - e^{-j3t}\right]u(t)$$

$$\boxed{y(t) = \left(\frac{1}{10}\sin(3t) + \frac{3}{10}e^{-t} - \frac{3}{10}\cos(3t)\right)u(t)}$$

6. Lathi 2.4-18

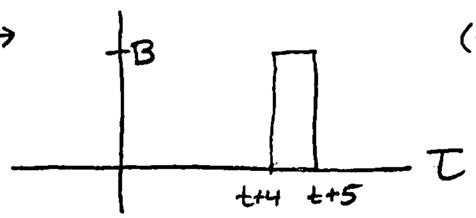
a)





$$y(t) = \int_{-\infty}^{\infty} x_1(\tau)\, x_2(t-\tau)\, d\tau$$

draw $x_2(t-\tau)$ as a function of τ $\rightarrow$



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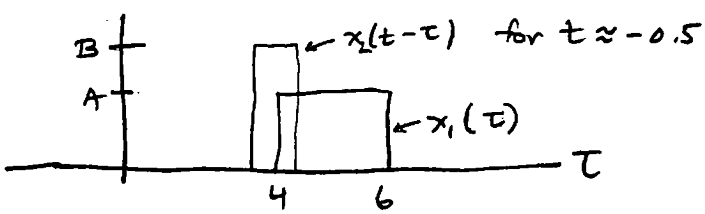
it is clear that there is no overlap until $t+5 \geq 4$ or, equivalently, $t \geq -1$

So, for $t < -1$, $y(t) = 0$

For $-1 \leq t \leq 0$ the area under the product of $x_1(\tau)\, x_2(t-\tau)$ is increasing linearly to a value of AB at $t=0$



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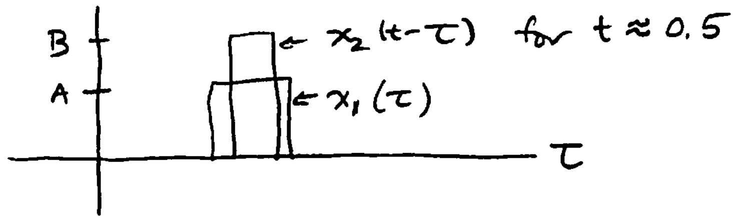
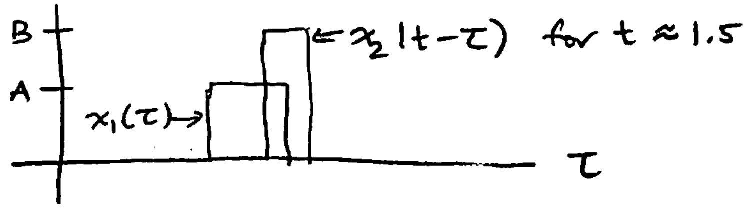
for $0 \le t \le 1$ the area under the product of $x_1(\tau)\,x_2(t-\tau)$ remains constant (equal to AB)

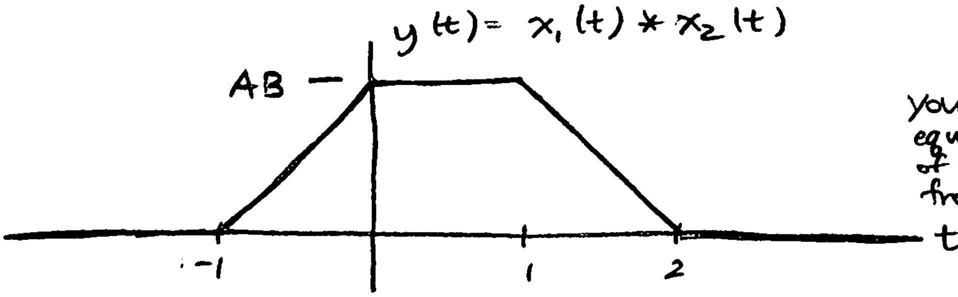


for $0 \le t \le 2$, the area under the product of $x_1(\tau)x_2(t-\tau)$ decreases from AB to zero linearly



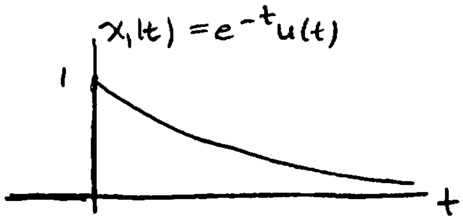
for $t > 2$, there is no overlap, hence the output is zero.

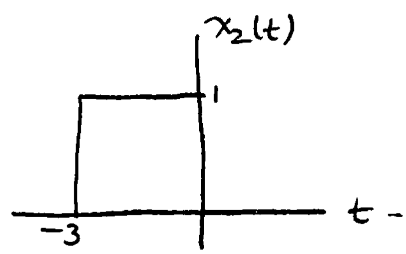
put it all together to get



you can write equations for each of these lines directly from the graph

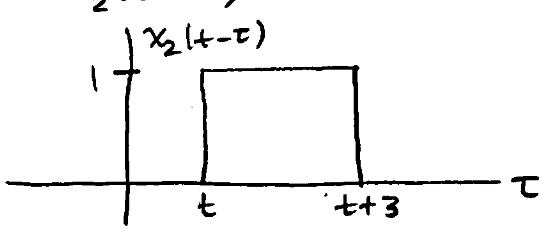
d)





$$y(t) = \int_{-\infty}^{\infty} x_1(\tau)\,x_2(t-\tau)\,d\tau$$

draw $x_2(t-\tau)$ as a function of τ



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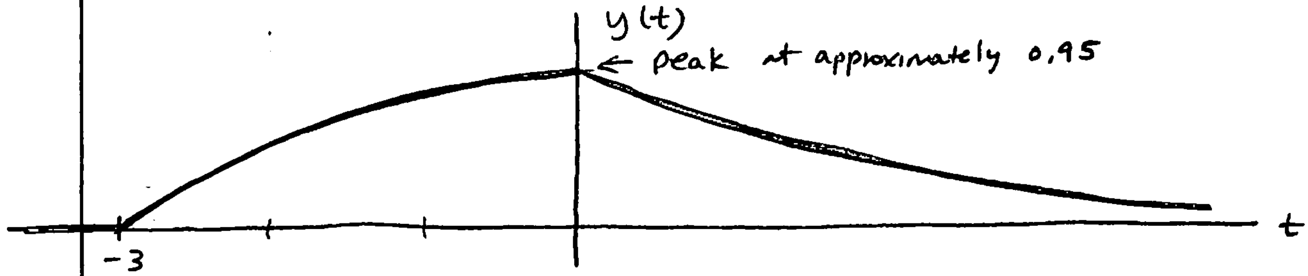
continued..

it should be clear that there is no overlap until $t+3 \geq 0$ or, equivalently, $t \geq -3$

as t increases from -3, the area under the curve increases, but by a little less each time. When t becomes zero, the area under the product $x_1(\tau)x_2(t-\tau)$ will peak and then decrease for all remaining time as $t \to \infty$.

So we can sketch the result



We can compute this output analytically using the fact that

$$x(t) = u(t+3) - u(t)$$

From the lecture, we know that

$$u(t) \longrightarrow \boxed{e^{-t}u(t)} \longrightarrow (1-e^{-t})u(t) \quad = y_1(t)$$

and time invariance implies

$$u(t+3) \longrightarrow \boxed{e^{-t}u(t)} \longrightarrow \left(1-e^{-(t+3)}\right)u(t+3) = y_2(t)$$

so $y(t) = y_2(t) - y_1(t) = \left(1-e^{-(t+3)}\right)u(t+3) - (1-e^{-t})u(t)$

$$y(t) = \begin{cases} 0 & \text{when } t < -3 \\ 1-e^{-(t+3)} & \text{when } -3 \leq t < 0 \\ e^{-t} - e^{-(t+3)} & \text{when } t \geq 0 \end{cases}$$

you can plot this in Matlab to confirm the sketch.

(end).