1.

(b) $T_0 = 10\pi$, $\omega_0 = \frac{2\pi}{T_0} = \frac{1}{5}$. Because of even symmetry, all the sine terms are zero.

$$\begin{aligned}
x(t) &= a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n}{5}t\right) + b_n \sin\left(\frac{n}{5}t\right) \\
a_0 &= \frac{1}{5} \quad \text{(by inspection)} \quad \text{(average value)} \\
a_n &= \frac{2}{10\pi}\int_{-\pi}^{\pi} \cos\left(\frac{n}{5}t\right) dt = \frac{1}{5\pi}\left(\frac{5}{n}\right)\sin\left(\frac{n}{5}t\right)\Big|_{-\pi}^{\pi} = \frac{2}{\pi n}\sin\left(\frac{n\pi}{5}\right) \\
b_n &= \frac{2}{10\pi}\int_{-\pi}^{\pi} \sin\left(\frac{n}{5}t\right) dt = 0 \quad \text{(integrand is an odd function of } t\text{)}
\end{aligned}$$

Here $b_n = 0$, and we allow C_n to take negative values. Note that $C_n = a_n$ for $n = 0, 1, 2, 3, \cdots$.

See C_n plot on following page $\theta_n = 0$ for all n.

(d) $T_0 = \pi$, $\omega_0 = 2$ and $x(t) = \frac{4}{\pi}t$. $a_0 = 0$ (by inspection). $a_n = 0$ $(n > 0)$ because of odd symmetry.

$$b_n = \frac{4}{\pi}\int_0^{\pi/4} \frac{4}{\pi} t \sin 2nt\, dt = \frac{2}{\pi n}\left(\frac{2}{\pi n}\sin\frac{\pi n}{2} - \cos\frac{\pi n}{2}\right)$$

$$\begin{aligned}
x(t) &= \frac{4}{\pi^2}\sin 2t + \frac{1}{\pi}\sin 4t - \frac{4}{9\pi^2}\sin 6t - \frac{1}{2\pi}\sin 8t + \cdots \\
&= \frac{4}{\pi^2}\cos\left(2t - \frac{\pi}{2}\right) + \frac{1}{\pi}\cos\left(4t - \frac{\pi}{2}\right) + \frac{4}{9\pi^2}\cos\left(6t + \frac{\pi}{2}\right) + \frac{1}{\pi}\cos\left(8t + \frac{\pi}{2}\right) + \cdot
\end{aligned}$$

See C_n and θ_n plots on following pages.

(e) $T_0 = 3$, $\omega_0 = 2\pi/3$.

$$a_0 = \frac{1}{3}\int_0^1 t\, dt = \frac{1}{6}$$

$$a_n = \frac{2}{3}\int_0^1 t\cos\frac{2n\pi}{3}t\,dt = \frac{3}{2\pi^2 n^2}\left[\cos\frac{2\pi n}{3} + \frac{2\pi n}{3}\sin\frac{2\pi n}{3} - 1\right]$$

$$b_n = \frac{2}{3}\int_0^1 t\sin\frac{2n\pi}{3}t\,dt = \frac{3}{2\pi^2 n^2}\left[\sin\frac{2\pi n}{3} - \frac{2\pi n}{3}\cos\frac{2\pi n}{3}\right]$$

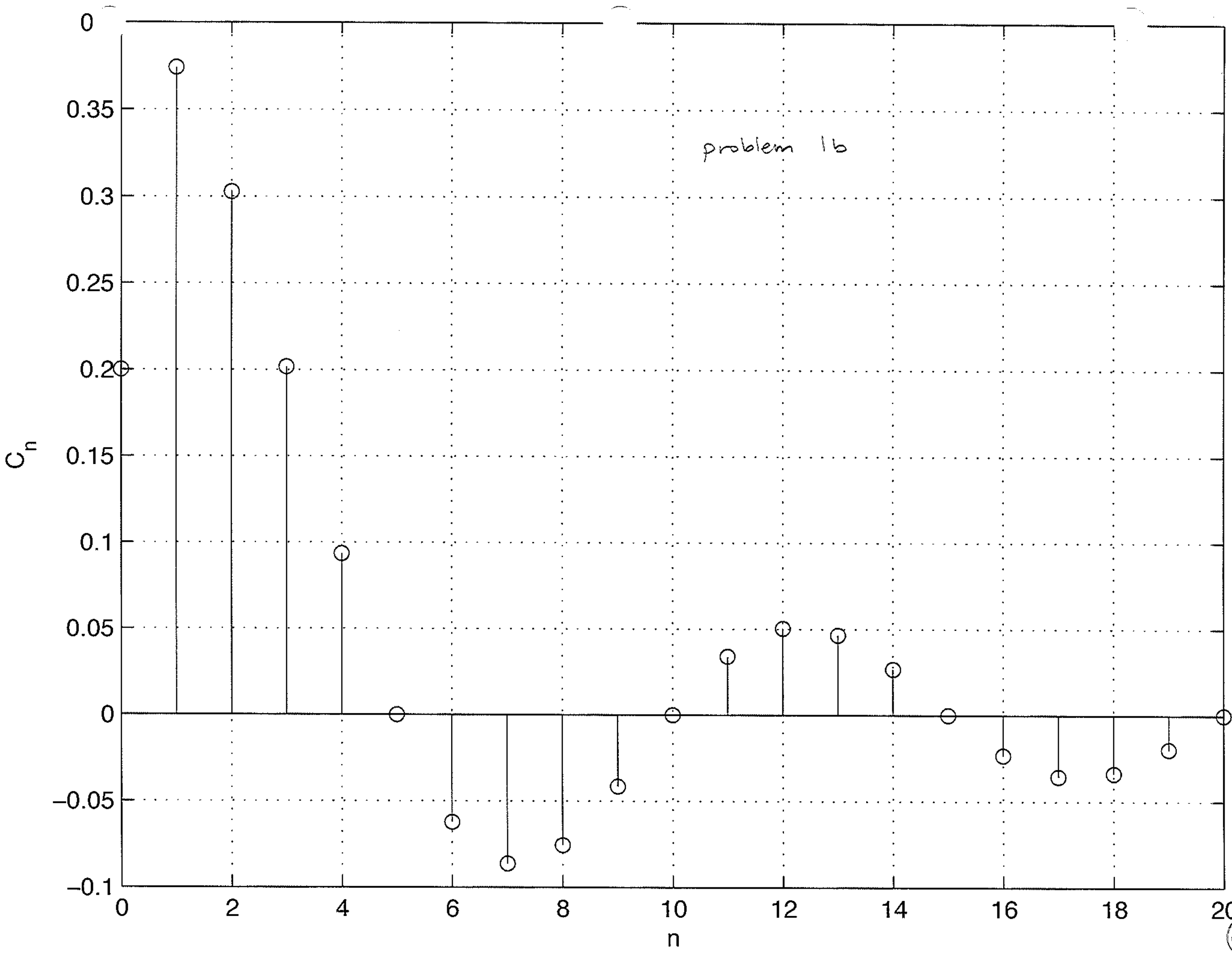
Therefore $C_0 = \frac{1}{6}$ and

$$C_n = \frac{3}{2\pi^2 n^2}\left[\sqrt{2 + \frac{4\pi^2 n^2}{9} - 2\cos\frac{2\pi n}{3} - \frac{4\pi n}{3}\sin\frac{2\pi n}{3}}\right]$$

and

$$\theta_n = \tan^{-1}\left(\frac{\frac{2\pi n}{3}\cos\frac{2\pi n}{3} - \sin\frac{2\pi n}{3}}{\cos\frac{2\pi n}{3} + \frac{2\pi n}{3}\sin\frac{2\pi n}{3} - 1}\right)$$

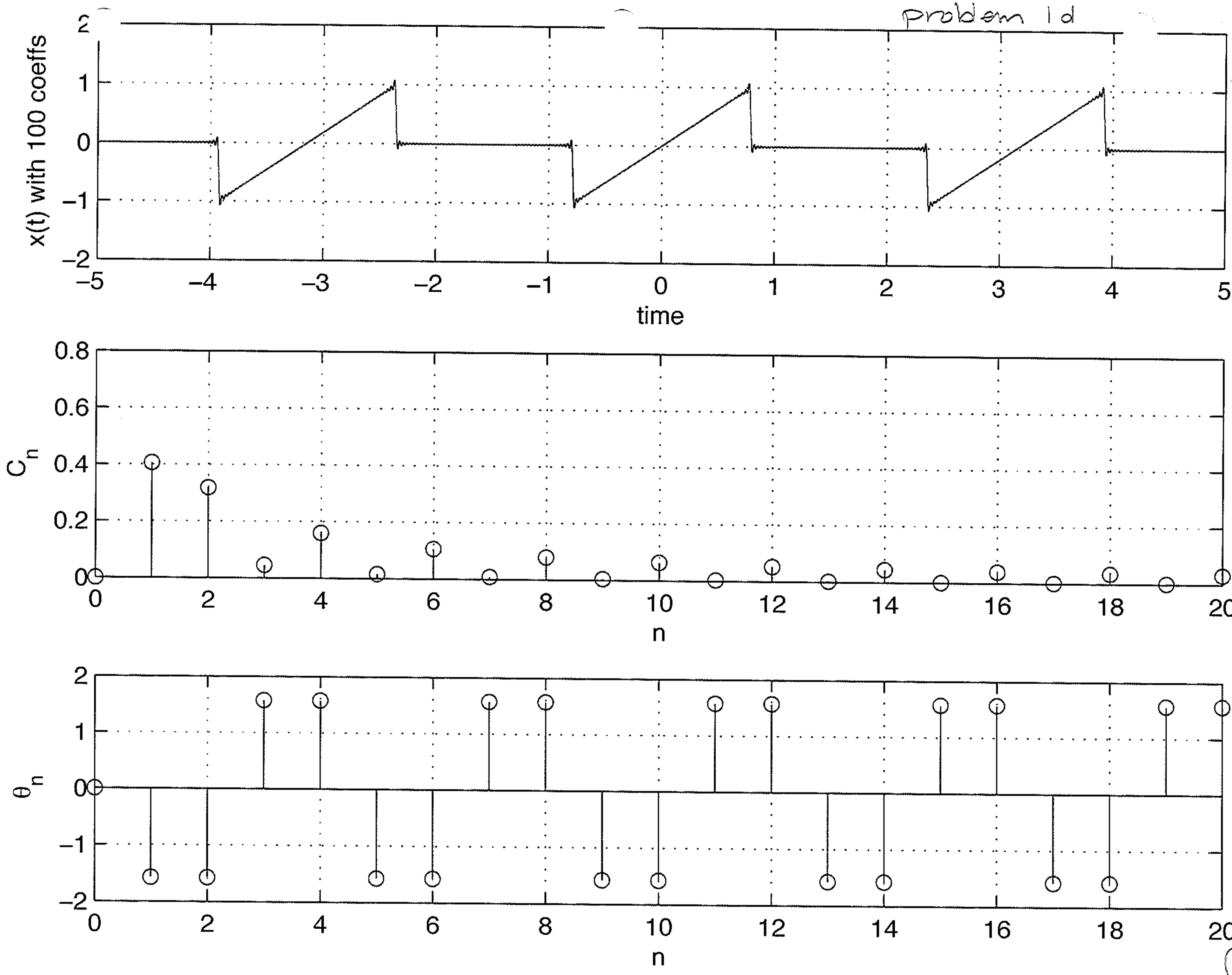
see C_n and θ_n plots on following pages.



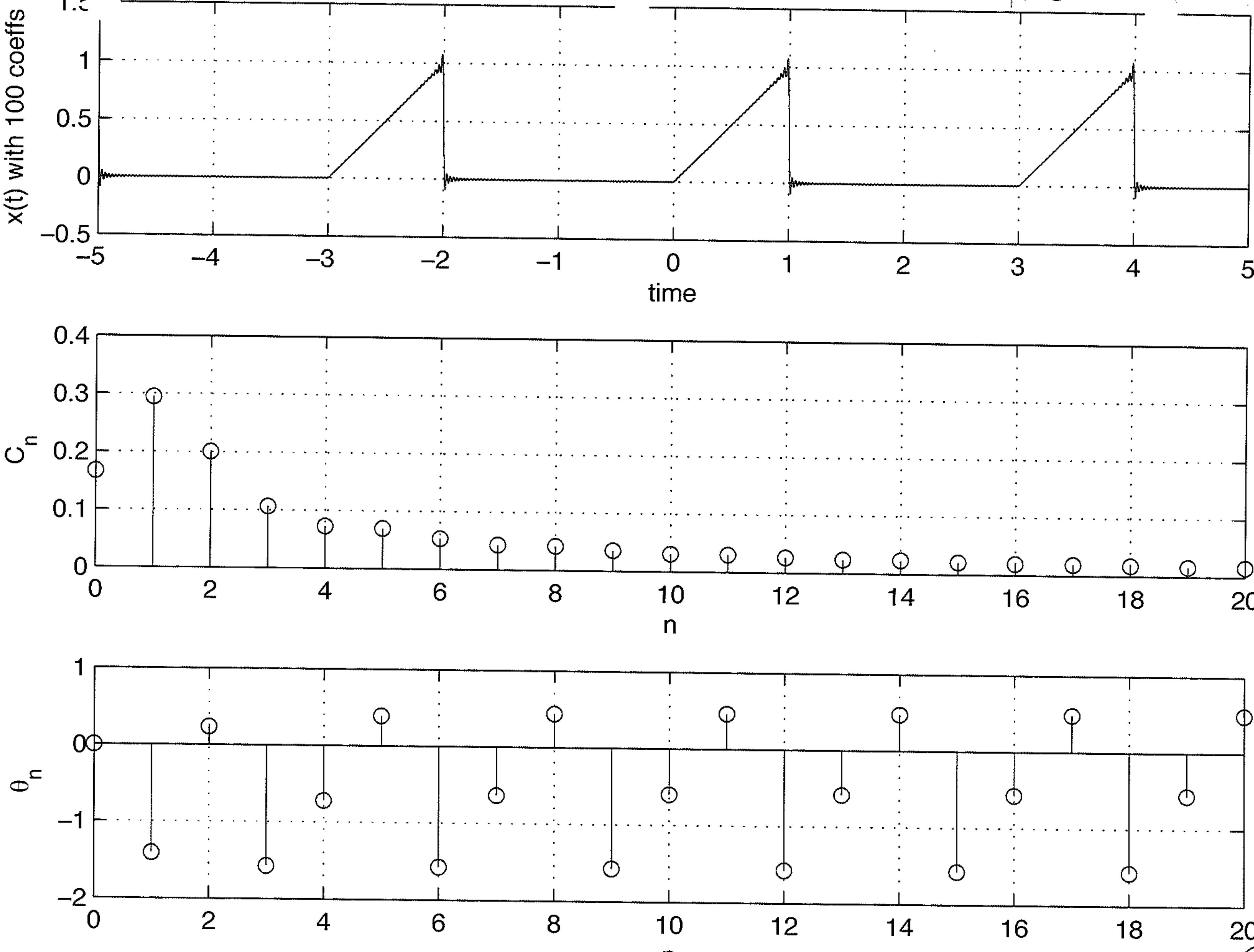
problem 1b
C_n
n
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
−0.05
−0.1
0 2 4 6 8 10 12 14 16 18 20

2

Problem 1d



prob 1e



x(t) with 100 coeffs
1.5
1
0.5
0
−0.5
−5
−4
−3
−2
−1
0
1
2
3
4
5
time
C_n
0.4
0.3
0.2
0.1
0
n
θ_n
1
0
−1
−2
0
2
4
6
8
10
12
14
16
18
20
n

4

2. 6.1-5. (a) For half wave symmetry

$$x(t) = -x\left(t \pm \frac{T_0}{2}\right)$$

and

$$a_n = \frac{2}{T_0}\int_0^{T_0} x(t)\cos n\omega_0 t\,dt = \frac{2}{T_0}\int_0^{T_0/2} x(t)\cos n\omega_0 t\,dt + \int_{T_0/2}^{T_0} x(t)\cos n\omega_0 t\,dt$$

Let $\tau = t - T_0/2$ in the second integral. This gives

$$\begin{aligned} a_n &= \frac{2}{T_0}\left[\int_0^{T_0/2} x(t)\cos n\omega_0 t\,dt + \int_0^{T_0/2} x\left(\tau + \frac{T_0}{2}\right)\cos n\omega_0\left(\tau + \frac{T_0}{2}\right)d\tau\right] \\ &= \frac{2}{T_0}\left[\int_0^{T_0/2} x(t)\cos n\omega_0 t\,dt + \int_0^{T_0/2} -x(\tau)[-\cos n\omega_0\tau]\,d\tau\right] \\ &= \frac{4}{T_0}\left[\int_0^{T_0/2} x(t)\cos n\omega_0 t\,dt\right] \end{aligned}$$

In a similar way we can show that

$$b_n = \frac{4}{T_0}\int_0^{T_0/2} x(t)\sin n\omega_0 t\,dt$$

To see that the even harmonics must be zero, note that

$$x\left(t - \frac{T_0}{2}\right) = a_0 + \sum_{n=1}^{\infty} a_n\cos(n\omega_0 t - n\pi) + b_n\sin(n\omega_0 t - n\pi) = -x(t)$$

By matching frequencies, we can see that $a_0 = -a_0 \Rightarrow a_0 = 0$

$$\left.\begin{aligned} &a_n\cos(n\omega_0 t - n\pi) \underset{\uparrow \text{when } n \text{ is even}}{=} a_n\cos(n\omega_0 t) = -a_n\cos(n\omega_0 t) \Rightarrow a_n = 0 \\ &b_n\sin(n\omega_0 t - n\pi) \underset{\uparrow \text{when } n \text{ is even}}{=} b_n\sin(n\omega_0 t) = -b_n\sin(n\omega_0 t) \Rightarrow b_n = 0 \end{aligned}\right\} \text{When } n \text{ is even}$$

Hence the even harmonics must all be zero.

(b) (i) $T_0 = 8$, $\omega_0 = \frac{\pi}{4}$, $a_0 = 0$ (by inspection). Half wave symmetry. Hence

$$\begin{aligned} a_n &= \frac{4}{8}\left[\int_0^4 x(t)\cos\frac{n\pi}{4}t\,dt\right] = \frac{1}{2}\left[\int_0^2 \frac{t}{2}\cos\frac{n\pi}{4}t\,dt\right] \\ &= \frac{4}{n^2\pi^2}\left(\cos\frac{n\pi}{2} + \frac{n\pi}{2}\sin\frac{n\pi}{2} - 1\right) \quad (n \text{ odd}) \\ &= \frac{4}{n^2\pi^2}\left(\frac{n\pi}{2}\sin\frac{n\pi}{2} - 1\right) \quad (n \text{ odd}) \end{aligned}$$

Therefore

$$a_n = \begin{cases} \frac{4}{n^2\pi^2}\left(\frac{n\pi}{2} - 1\right) & n = 1, 5, 9, 13, \cdots \\ -\frac{4}{n^2\pi^2}\left(\frac{n\pi}{2} + 1\right) & n = 3, 7, 11, 15, \cdots \end{cases}$$

Similarly

$$b_n = \frac{1}{2}\int_0^2 \frac{t}{2}\sin\frac{n\pi}{4}t\,dt = \frac{4}{n^2\pi^2}\left(\sin\frac{n\pi}{2} - \frac{n\pi}{2}\cos\frac{n\pi}{2}\right) = \frac{4}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right) \quad (n \text{ odd})$$

and

$$x(t) = \sum_{n=1,3,5,\cdots}^{\infty} a_n\cos\frac{n\pi}{4}t + b_n\sin\frac{n\pi}{4}t$$

(ii) $T_0 = 2\pi$, $\omega_0 = 1$, $a_0 = 0$ (by inspection). Half wave symmetry. Hence

$$x(t) = \sum_{n=1,3,5,\cdots}^{\infty} a_n\cos nt + b_n\sin nt$$

$$
\begin{aligned}
a_n &= \frac{2}{\pi}\int_0^{\pi} e^{-t/10}\cos nt\,dt \\
&= \frac{2}{\pi}\left[\frac{e^{-t/10}}{n^2+0.01}(-0.1\cos nt + n\sin nt)\right]_0^{\pi} \quad (n \text{ odd}) \\
&= \frac{2}{\pi}\left[\frac{e^{-\pi/10}}{n^2+0.01}(0.1) - \frac{1}{n^2+0.01}(-0.1)\right] \\
&= \frac{2}{10\pi(n^2+0.01)}(e^{-\pi/10}-1) = \frac{0.0465}{n^2+0.01}
\end{aligned}
$$

and

$$
\begin{aligned}
b_n &= \frac{2}{\pi}\int_0^{\pi} e^{-t/10}\sin nt\,dt \\
&= \frac{2}{\pi}\left[\frac{e^{-t/10}}{n^2+0.01}(-0.1\sin nt - n\cos nt)\right]_0^{\pi} \quad (n \text{ odd}) \\
&= \frac{2n}{(n^2+0.01)}(e^{-\pi/10}-1) = \frac{1.461n}{n^2+0.01} \quad .
\end{aligned}
$$

3. a) this is a periodic signal with period $T_0 = 2\pi$

$3\sin(t+2\pi) + 2\sin(3(t+2\pi)) = 3\sin(t) + 2\sin(3t)$

hence $\omega_0 = 1$.

The series representation has the first and third harmonics.

b) this is also a periodic signal with period $T_0 = 2\pi$

$2 + 5\sin(4(t+2\pi)) + 4\cos(7(t+2\pi)) =$

$2 + 5\sin(4t) + 4\cos(7t)$

Hence $\omega_0 = 1$

The series representation has the 0th, 4th, and 7th harmonics.

h) $(3\sin(2t) + \sin(5t))^2 = 9\sin^2(2t) + 6\sin(2t)\sin(5t) + \sin^2(5t)$

use standard trig identities to write

$= \frac{9}{2}(1-\cos(4t)) + 3(\cos(3t) - \cos(7t)) + \frac{1}{2}(1-\cos(10t))$

This signal is clearly periodic with period $2\pi = T_0$

$\Rightarrow \omega_0 = 1$. The series representation has the 0th, 3rd, 4th, 7th and 10th harmonics.

4. From example 6.4, we have the Fourier coefficients for the input signal $x(t)$.

$$C_0 = \frac{1}{2}$$

$$C_n = \begin{cases} 0 & n \text{ even} \\ \frac{2}{\pi n} & n \text{ odd} \end{cases}$$

$$\theta_n = \begin{cases} 0 & \text{for all } n \neq 3, 7, 11, 15, \ldots \\ -\pi & \text{for } n = 3, 7, 11, 15, \ldots \end{cases}$$

Recall that the impedance of a capacitor is $\frac{1}{j\omega C}$.

Hence $$y(t) = \frac{\frac{1}{j\omega C}}{R + \frac{1}{j\omega C}} x(t) = \frac{1}{1 + j\omega RC} x(t)$$

when $x(t)$ is a sinusoidal input with frequency ω.

Set $R = C = 1$ to get $$\frac{1}{1 + j\omega} = H(\omega)$$

we want to evaluate the amplitude and phase of this expression for $\omega = 0, \omega_0, 2\omega_0, \ldots$

From example 6.4, we know $\omega_0 = 1$.

ω	$\lvert H(\omega) \rvert$	$\measuredangle H(\omega)$
$0 \cdot \omega_0$	1	0
$1 \cdot \omega_0$	$\frac{1}{\sqrt{2}}$	$-\pi/4$
$2 \cdot \omega_0$	$\frac{1}{\sqrt{5}}$	$-\tan^{-1}\left(\frac{2}{1}\right)$
etc.		

← this means that $C_0' = C_0$.

In general, we have $\lvert H(n\omega_0) \rvert \cdot \frac{1}{\sqrt{1+n^2}}$

and $\measuredangle H(n\omega_0) = \tan^{-1}(-n)$

Hence, at the output

$$C_n' = C_n H(n\omega_0) = \begin{cases} 0 & n \text{ even} \\ \frac{2}{\pi n \sqrt{1+n^2}} & n \text{ odd} \end{cases}$$

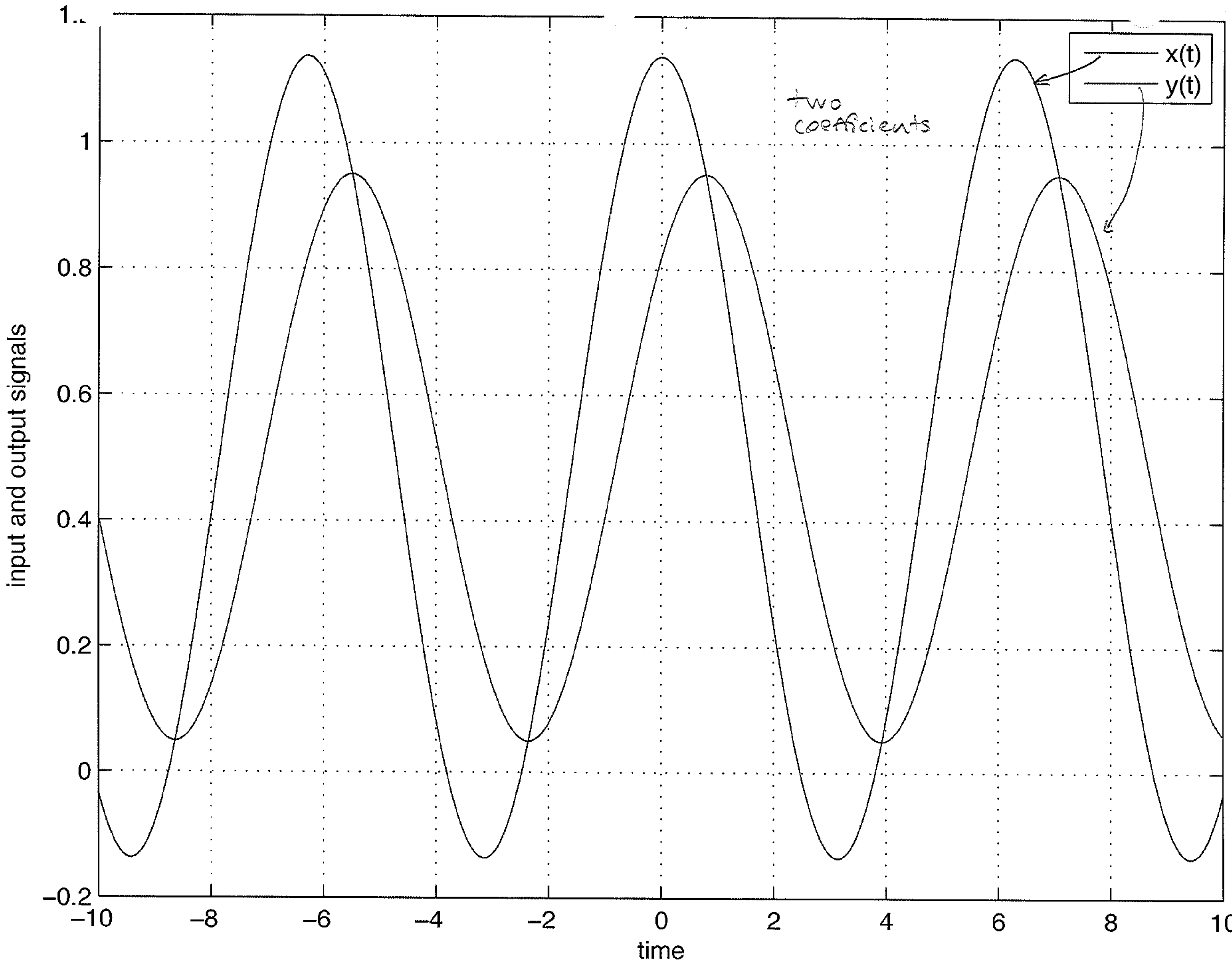
continued

and

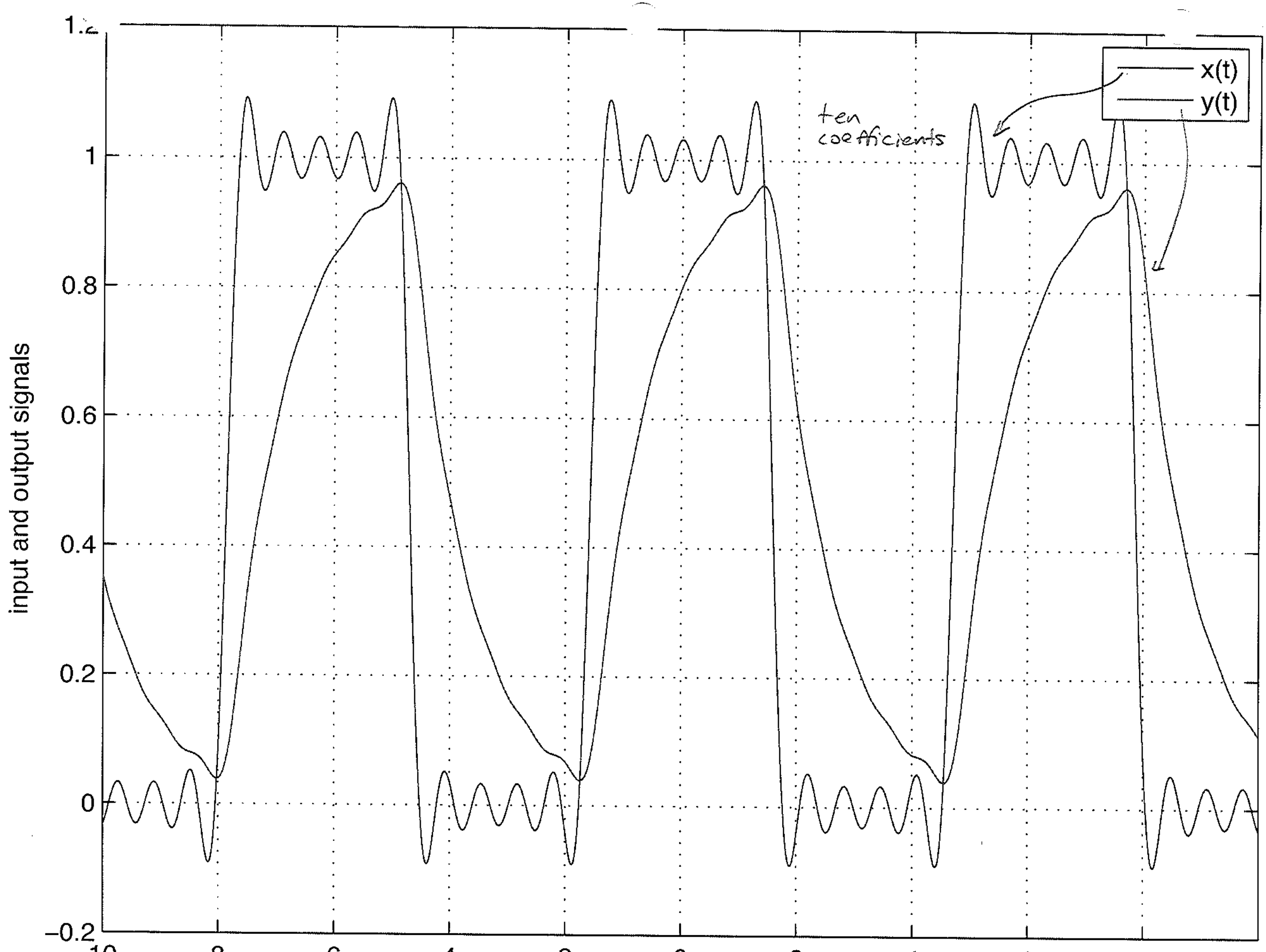
$$\theta_n' = \begin{cases} 0 + \tan^{-1}(-n) & \text{for all } n \neq 3, 7, 11, 15, \ldots \\ -\pi + \tan^{-1}(-n) & \text{for all } n = 3, 7, 11, 15, \ldots \end{cases}$$

See following pages for plots using first two, ten, one hundred coefficients.

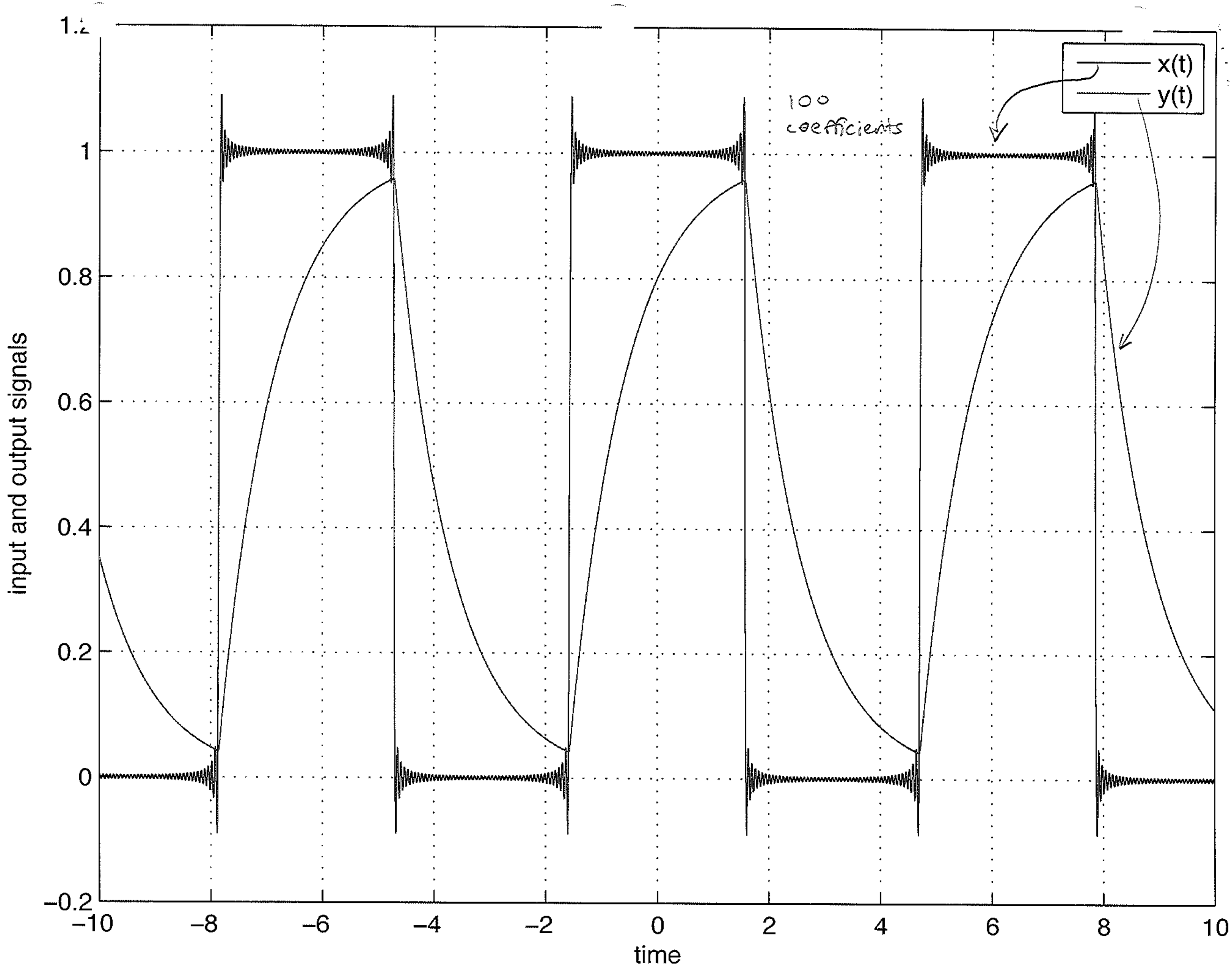
```
% ====================================================
% USER PARAMETERS
% ====================================================
N = 100;
w0 = 1;
t = -10:0.001:10;
C0 = 1/2;
% ====================================================

x = zeros(1,length(t))+C0;      % make array for input (add C0)
y = zeros(1,length(t))+C0;      % make array for output (add C0)

for n=1:N,

    % compute Cn for input and Cnprime for output
    if round(n/2)==(n/2),
        Cn = 0;
        Cnprime = 0;
    else
        Cn = 2/pi/n;
        Cnprime = Cn/sqrt(1+n^2);
    end

    % compute thetan for input and thetanprime for output
    if round((n-3)/4)==(n-3)/4,
        thetan = -pi;
        thetanprime = thetan+atan(-n);
    else
        thetan = 0;
        thetanprime = thetan+atan(-n);
    end

    % now add to signals
    x = x+Cn*cos(n*w0*t+thetan);
    y = y+Cnprime*cos(n*w0*t+thetanprime);

end

% plot
plot(t,x,t,y);
xlabel('time');
ylabel('input and output signals');
legend('x(t)','y(t)');
grid on
```


x(t)
y(t)
two
coefficients
1
0.8
0.6
0.4
0.2
0
−0.2
input and output signals
−10
−8
−6
−4
−2
0
2
4
6
8
10
time

9



x(t)
y(t)
ten coefficients
input and output signals
time



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