EE4304 C-term 2007: Lecture 14 Supplemental Slides

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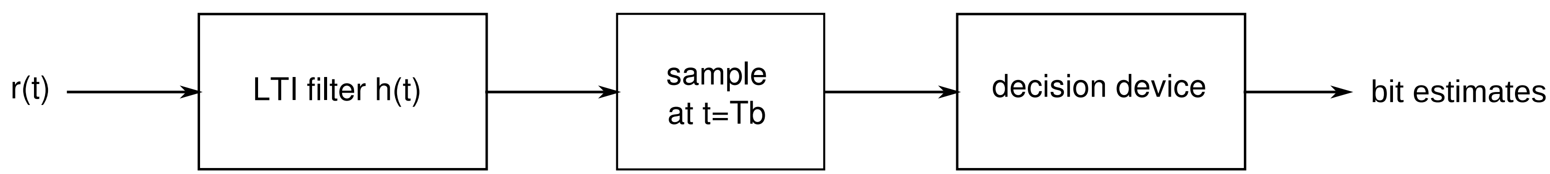
Baseband M-PAM Communications

The baseband M-PAM transmitter transmits waveforms from the "alphabet"

$$u(t) \in \{s_1(t), \ldots, s_M(t)\}$$

where $s_i(t) = a_i s(t)$. The waveforms are distinguished by their real-valued amplitudes a_i; $s(t)$ is the "root waveform" common to all transmissions in the M-PAM alphabet.

The baseband M-PAM receiver:



We have already optimized the linear filter at the front end of this reciever

$$h_{opt}(t) \quad = \quad ks(T_b - t)$$

The "matched filter" optimizes the SNR at the sampling instant $t = T_b$.

Optimizing the Decision Device

Focus on 2-PAM case first. Transmitter sends

$$\begin{aligned} \text{binary one} &\rightarrow a_1 s(t) \\ \text{binary zero} &\rightarrow a_2 s(t) \end{aligned}$$

We assume binary ones are sent with probability p and binary zeros are sent with probability $1-p$. We want to design the decision device to *minimize the probability of error*.

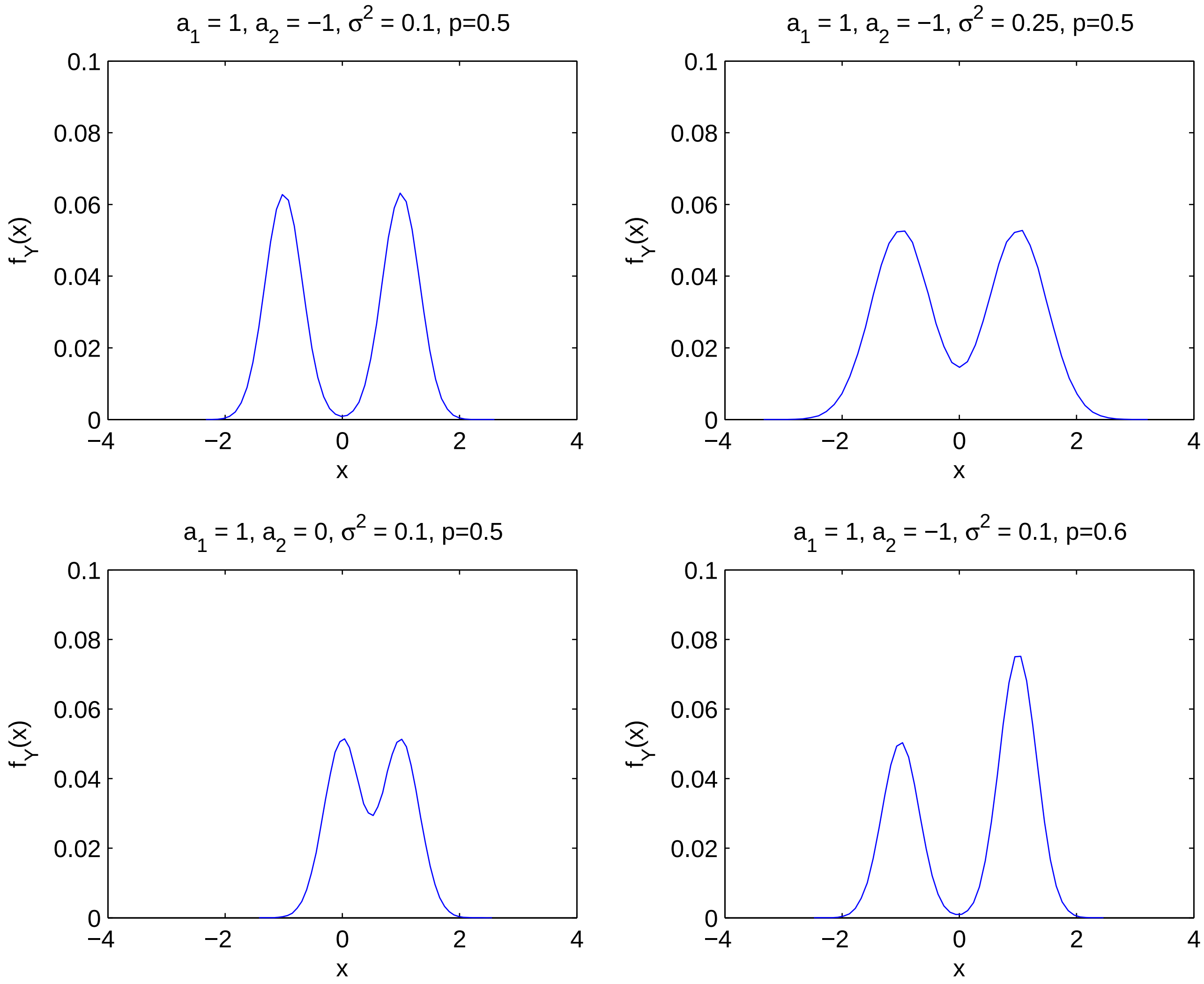
At the receiver, the input to the decision device is $Y = y(T_b)$. Y is a random variable! Assuming that $h(t) = s(T_b - t)$, we can write Y as

$$Y = \alpha a_i \underbrace{\int_0^{T_b} s^2(t)\, dt}_{\mathcal{E}} + N_0(T_b) = \alpha a_i \mathcal{E} + N_0(T_b)$$

where $N_0(t)$ is the filtered noise random process.

- What is random here?
- What does the pdf of Y look like?

Some Histograms of Y



Minimizing the Probability of Error (2-PAM)

Probability of making an error:

$$P_e \quad = \quad P(0 \text{ decided}, 1 \text{ sent}) + P(1 \text{ decided}, 0 \text{ sent})$$

Recall $P(A, B)$ is the probability that *both* event A and event B occur. We would like to use "Bayes rule" to rewrite the error probability in a form that is easier to analyze. Bayes rule states

$$P(A, B) \quad = \quad P(B)P(A|B)$$

where $P(A|B)$ is the probability of event A *given* event B occured. Applying this to our problem, we can write

$$P_e \quad = \quad P(1 \text{ sent})P(0 \text{ decided} \,|\, 1 \text{ sent}) + P(0 \text{ sent})P(1 \text{ decided} \,|\, 0 \text{ sent})$$

Look at the case "1 sent". We can write

$$Y_1 \quad = \quad \alpha a_1 \mathcal{E} + N_0(T_b)$$

What is random here? What can we say about Y_1?

Conditional Distributions of Y_1 and Y_0

If the noise in the communication system is AWGN with PSD $S_N(f) = \frac{N_0}{2} \ \forall f$, we can say a lot about the random variable $N_0(T_b)$.

- Since the filter is LTI, the filtered random process $N_0(t)$ is a Gaussian r.p.
- Sampling a Guassian r.p. yields a Gaussian random variable. Hence, $N_0(T_b)$ has a Gaussian pdf.
- What is the mean?
- What is the variance?

These results imply that, given "1 sent", we can say that

$$Y_1 \sim \mathcal{N}\left(\alpha a_1 \mathcal{E}, \frac{N_0 \mathcal{E}}{2}\right) \text{ (distribution of } Y \text{ given "1 sent")}$$

Same analysis applies for the case when "0 sent".

$$Y_0 \sim \mathcal{N}\left(\alpha a_2 \mathcal{E}, \frac{N_0 \mathcal{E}}{2}\right) \text{ (distribution of } Y \text{ given "0 sent")}$$

Minimizing the Probability of Error (2-PAM)

Assuming $a_1 > a_2$, the decision device will make decisions according the following rule

$$\text{decide binary one if } Y \geq \lambda$$
$$\text{decide binary zero if } Y < \lambda$$

where λ is a threshold that we pick to *minimize the probability of error*.

Manipulating our expression for probability of error, we can write

$$\begin{aligned}
P_e &= P(1\text{ sent})P(0\text{ decided}\,|\,1\text{ sent}) + P(0\text{ sent})P(1\text{ decided}\,|\,0\text{ sent}) \\
P_e &= P(1\text{ sent})P(Y < \lambda, |\,1\text{ sent}) + P(0\text{ sent})P(Y \geq \lambda\,|\,0\text{ sent}) \\
P_e &= P(1\text{ sent})P(Y_1 < \lambda) + P(0\text{ sent})P(Y_2 \geq \lambda) \\
P_e &= pP(Y_1 < \lambda) + (1-p)P(Y_2 \geq \lambda)
\end{aligned}$$

The Q-Function

Recall that the pdf of a Gaussian random variable distributed as $X \sim N(\mu, \sigma^2)$ can be written as

$$f_X(x) \quad = \quad \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2}$$

In order to have a nice compact form for writing out the error probability expression, we need to define the "Q-function". Suppose you have a random variable $X \sim \mathcal{N}(0, 1)$. Define

$$Q(x) = P(X > x) = \int_x^\infty \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\, dt$$

Note that

$$\begin{aligned}
Q(x) &= P(X < -x) \\
Q(x) &= 1 - P(X > -x) = 1 - Q(-x) \\
Q(x) &= \frac{1}{2}\text{erfc}\left(\frac{x}{\sqrt{2}}\right) \\
\text{erfc}(x) &= 2Q(\sqrt{2}x)
\end{aligned}$$

2-PAM Probability of Error

Look at

$$P(Y_1 < \lambda) = \frac{1}{\sqrt{2\pi}\sqrt{\frac{N_0\mathcal{E}}{2}}} \int_{-\infty}^{\lambda} \exp\left(\frac{-(x - \alpha a_1 \mathcal{E})^2}{2\frac{N_0\mathcal{E}}{2}} \right) dx$$

Variable substitution

$$t = \frac{x - \alpha a_1 \mathcal{E}}{\sqrt{\frac{N_0\mathcal{E}}{2}}}$$

Rewrite

$$P(Y_1 < \lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{v} e^{-t^2/2}\, dt$$

where

$$v = \frac{\lambda - \alpha a_1 \mathcal{E}}{\sqrt{\frac{N_0\mathcal{E}}{2}}}$$

2-PAM Probability of Error

Hence

$$P(Y_1 < \lambda) = Q\left(-v\right) = Q\left(\frac{\alpha a_1 \mathcal{E} - \lambda}{\sqrt{\frac{N_0 \mathcal{E}}{2}}}\right)$$

Similarly, we can compute

$$P(Y_0 \geq \lambda) = Q\left(\frac{\lambda - \alpha a_2 \mathcal{E}}{\sqrt{\frac{N_0 \mathcal{E}}{2}}}\right)$$

Put it all together...

$$\begin{aligned} P_e &= pP(Y_1 < \lambda) + (1-p)P(Y_2 \geq \lambda) \\ P_e &= pQ\left(\frac{\alpha a_1 \mathcal{E} - \lambda}{\sqrt{\frac{N_0 \mathcal{E}}{2}}}\right) + (1-p)Q\left(\frac{\lambda - \alpha a_2 \mathcal{E}}{\sqrt{\frac{N_0 \mathcal{E}}{2}}}\right) \end{aligned}$$

Probability of error clearly depends on λ. What choice of λ minimizes P_e?

Optimal Choice of λ for 2-PAM

To find optimal threshold λ, we set $\frac{\partial P_e}{\partial \lambda} = 0$ and solve for λ. This is done in Haykin p.257. The main result is

$$\lambda_{opt} = \frac{\alpha \mathcal{E}}{2}(a_1 + a_2) + \frac{N_0}{2\alpha(a_1 - a_2)} \ln\left(\frac{1-p}{p}\right)$$

Special cases:

- When $p = 0.5$, $\lambda_{opt} = \frac{\alpha \mathcal{E}}{2}(a_1 + a_2)$.
- When $p = 0.5$ and $a_1 = -a_2$, $\lambda_{opt} = 0$.

$$P_e = \frac{1}{2} Q\left(\frac{\alpha a_1 \mathcal{E}}{\sqrt{\frac{N_0 \mathcal{E}}{2}}}\right) + \frac{1}{2} Q\left(\frac{-\alpha a_2 \mathcal{E}}{\sqrt{\frac{N_0 \mathcal{E}}{2}}}\right) = Q\left(\frac{\alpha a_1 \mathcal{E}}{\sqrt{\frac{N_0 \mathcal{E}}{2}}}\right)$$

- When $p > 0.5$, λ_{opt} decreases (more likely to decide 1 sent).
- When $p < 0.5$, λ_{opt} increases (more likely to decide 0 sent).
- When $p = 1$, $\lambda_{opt} = -\infty$ (always decide 1 sent).
- When $p = 0$, $\lambda_{opt} = +\infty$ (always decide 0 sent).

Remarks

This sort of analysis becomes difficult for communication systems with larger alphabets, e.g. 4-PAM, 8-PAM, 1024-QAM.

We are going to do two things to facilitate analysis of these systems:

1. From now on, we will assume that all of the symbols in the transmitter's alphabet

$$\mathcal{S} = \{s_1(t), \ldots, s_M(t)\}$$

   are sent with equal probability. This is almost always the case in real communication systems.

2. We are going to develop the idea of "geometric representation" of digital communication systems
   - Generalize our error probability analysis to more complicated communication formats
   - Simplify and reduce notation
   - Eliminate integrals!
   - Provide more insight into performance and design tradeoffs

Geometric Representation: Main Idea

Given a digital communication system with and alphabet containing M waveforms

$$\mathcal{S} = \{s_1(t), \ldots, s_M(t)\} \text{ for } t \in [0, T_b),$$

we want want to find an *orthonormal basis* containing $N \leq M$ waveforms

$$\{\psi_1(t), \ldots, \psi_N(t)\} \text{ for } t \in [0, T_b),$$

such that

$$\int_0^{T_b} \psi_n^2(t)\, dt = 1 \text{ for all } n \in \{1, \ldots, N\} \text{ and } \int_0^{T_b} \psi_m(t)\psi_n(t)\, dt = 0 \text{ for all } n \neq m$$

and

$$\begin{aligned} s_1(t) &= s_{1,1}\psi_1(t) + \cdots + s_{1,N}\psi_N(t) \\ &\vdots \\ s_M(t) &= s_{M,1}\psi_1(t) + \cdots + s_{M,N}\psi_N(t) \end{aligned}$$

where $s_{m,n}$ is a constant.

Geometric Representation: Why do this?

If we can find this "orthonormal basis", we can represent each waveform $s_m(t)$, $M \in \{1, \ldots, M\}$, as an N-dimensional vector, i.e.

waveform		vector representation
$s_1(t)$	$\Leftrightarrow$	$[s_{1,1}, \ldots, s_{1,N}]$
$\vdots$	$\vdots$	$\vdots$
$s_M(t)$	$\Leftrightarrow$	$[s_{M,1}, \ldots, s_{M,N}]$

Analyzing waveforms requires lots of calculus. Analyzing vectors requires lots of geometry. Which do you prefer?

Vector notation: Let $\boldsymbol{s}_m = [s_{m,1}, \ldots, s_{m,N}]^\top$ and $\boldsymbol{\psi}(t) = [\psi_1(t), \ldots, \psi_N(t)]^\top$. The waveform $s_m(t)$ can be compactly written as $s_m(t) = \boldsymbol{s}_m^\top \boldsymbol{\psi}(t)$.

For more information, see Chapter 5 of Haykin.

Gram-Schmidt Orthogonalization Procedure - Step 1

We need a procedure to find a suitable orthonormal basis for our alphabet $\mathcal{S}$. One valid procedure is called the "Gram-Schmidt Orthogonalization Procedure".

Compute

$$\mathcal{E}_1 = \int_0^{T_b} s_1^2(t)\,dt$$

Set

$$\psi_1(t) = \frac{s_1(t)}{\sqrt{\mathcal{E}_1}}$$

It is easy to show that $\int_0^{T_b} \psi_1^2(t)\,dt = 1$. We now have our first basis function.

Gram-Schmidt Orthogonalization Procedure - Step 2

Compute

$$c_{2,1} = \int_0^{T_b} s_2(t)\psi_1(t)\,dt$$

Compute

$$d_2(t) = s_2(t) - c_{2,1}\psi_1(t)$$

If $d_2(t) = 0$, go to step 3. Otherwise, compute

$$\mathcal{E}_2 = \int_0^{T_b} d_2^2(t)\,dt$$

Set

$$\psi_2(t) = \frac{d_2(t)}{\sqrt{\mathcal{E}_2}}$$

It is easy to show that $\int_0^{T_b} \psi_2^2(t)\,dt = 1$ and also fairly straightforward to show that $\int_0^{T_b} \psi_1(t)\psi_2(t)\,dt = 0$. We now have our second basis function.

Gram-Schmidt Orthogonalization Procedure - Step 3

Compute

$$c_{3,1} = \int_0^{T_b} s_3(t)\psi_1(t)\,dt \text{ and } c_{3,2} = \int_0^{T_b} s_3(t)\psi_2(t)\,dt$$

Compute

$$d_3(t) = s_3(t) - c_{3,1}\psi_1(t) - c_{3,2}\psi_2(t)$$

If $d_3(t) = 0$, go to step 4. Otherwise, compute

$$\mathcal{E}_3 = \int_0^{T_b} d_3^2(t)\,dt$$

Set

$$\psi_3(t) = \frac{d_3(t)}{\sqrt{\mathcal{E}_3}}$$

It is easy to show that $\int_0^{T_b} \psi_3^2(t)\,dt = 1$. Proving orthogonality to $\psi_1(t)$ and $\psi_2(t)$ is starting to get tedious, but it can be done. We now have our third basis function.

Gram-Schmidt Orthogonalization Procedure-Step m

Compute

$$c_{m,k} = \int_0^{T_b} s_m(t)\psi_k(t)\,dt \text{ for all } k \in \{1, \ldots, m-1\}$$

Compute

$$d_m(t) = s_m(t) - \sum_{k=1}^{m-1} c_{m,k}\psi_k(t)$$

If $d_m(t) = 0$, go to step $m+1$. Otherwise, compute

$$\mathcal{E}_m = \int_0^{T_b} d_m^2(t)\,dt$$

Set

$$\psi_m(t) = \frac{d_m(t)}{\sqrt{\mathcal{E}_m}}$$

Keep doing this up to (and including) $m = M$.

Intuition and Conclusions (I)

- The basis will not be unique. For example, you could reorder the signal set $\mathcal{S}$ and you would get a different basis.
- The number of functions in the basis (N) is unique. Note that $1 \leq N \leq M$.
- The step

$$c_{m,k} = \int_0^{T_b} s_m(t)\psi_k(t)\,dt \text{ for all } k \in \{1, \ldots, m-1\}$$

actually computes the *projection* of the signal $s_m(t)$ on each of the previously computed basis functions. If $s_m(t)$ can be expressed completely by these basis functions, that is

$$s_m(t) = \sum_{k=1}^{m-1} c_{m,k}\psi_k(t),$$

then no new basis function will be created from $s_m(t)$ (because $d_m(t) = 0$).

Intuition and Conclusions (II)

- The step

$$d_m(t) = s_m(t) - \sum_{k=1}^{m-1} c_{m,k}\psi_k(t)$$

  computes the *residual* part of $s_m(t)$ that cannot be expressed as a sum of the previously computed basis functions. You can show that this residual is in fact *orthogonal* to all of the previously computed basis functions.

- The final steps just normalize this residual so that the basis is *orthonormal*.