EE4304 C-term 2007: Lecture 17 Supplemental Slides

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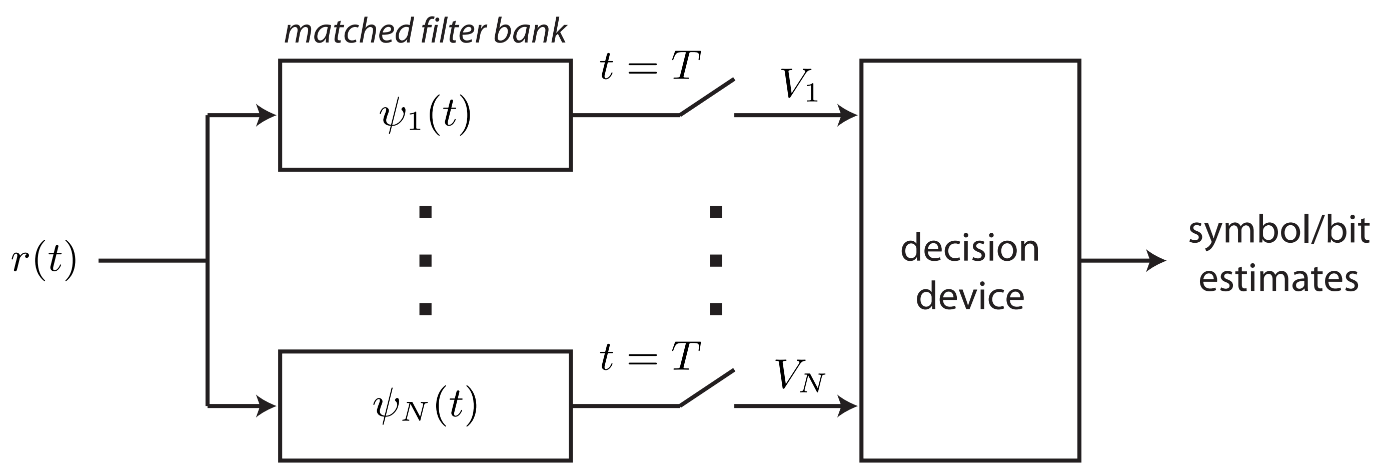
Geometric Representation: Optimal Receiver

Geometric representation of a general digital communication system alphabet:

$$
\begin{aligned}
s_1(t) &= c_{1,1}\psi_1(t) + \cdots + c_{1,N}\psi_N(t) \\
\vdots \quad & \vdots \quad \vdots \\
s_M(t) &= c_{M,1}\psi_1(t) + \cdots + c_{M,N}\psi_N(t)
\end{aligned}
$$

The optimal receiver uses an N-dimensional bank of matched filters:



This receiver maximizes the SNR at time $t = T$ in all N dimensions.

Analysis of the Matched Filter Bank Receiver - 1

What does V_n look like?

$$\begin{aligned} V_n &= \int_0^T r(t)\psi_n(t)\,dt \\ &= \alpha \underbrace{\int_0^T s_m(t)\psi_n(t)\,dt}_{\text{this is just the coordinate } c_{m,n}} + \underbrace{\int_0^T N(t)\psi_n(t)\,dt}_{\text{call this } w_n} \\ &= \alpha c_{m,n} + w_n \end{aligned}$$

Clearly V_n is a random variable. In fact,

- $c_{m,n}$ is random since we don't know which symbol was sent (m is unknown)
- w_n is random since the filtered, sampled noise is random

Vector notation

$$\boldsymbol{V} = \begin{bmatrix} V_1 \\ \vdots \\ V_N \end{bmatrix} = \alpha \begin{bmatrix} c_{m,1} \\ \vdots \\ c_{m,N} \end{bmatrix} + \begin{bmatrix} w_1 \\ \vdots \\ w_N \end{bmatrix} = \alpha \boldsymbol{c}_m + \boldsymbol{W}$$

Analysis of the Matched Filter Bank Receiver - 2

What can we say about the noise vector $\boldsymbol{W}$?

- Each w_n is Gaussian distributed (AWGN $\rightarrow$ LTI filter $\rightarrow$ sample $\rightarrow$ GRV)
- $\mathsf{E}[w_n] = 0$ (AWGN $\rightarrow$ LTI filter $\rightarrow$ zero-mean GRP $\rightarrow$ zero-mean GRV)
- Variance

$$\begin{aligned}
\mathsf{var}[w_n] &= \mathsf{E}[w_n^2] \\
&= \mathsf{E}\left[\left(\int_0^T N(t)\psi_n(t)\,dt\right)^2\right] \\
&= \int_0^T \int_0^T \underbrace{\mathsf{E}[N(t_1)N(t_2)]}_{=\frac{N_0}{2}\delta(t_1-t_2)\text{ since white noise}} \psi_n(t_1)\psi_n(t_2)\,dt_1 dt_2 \\
&= \frac{N_0}{2}\int_0^T \psi_n^2(t)\,dt \\
&= \frac{N_0}{2} \text{ for all } n \text{ since the basis functions are normal}
\end{aligned}$$

Analysis of the Matched Filter Bank Receiver - 3

We know everything about w_n. Is this enough?

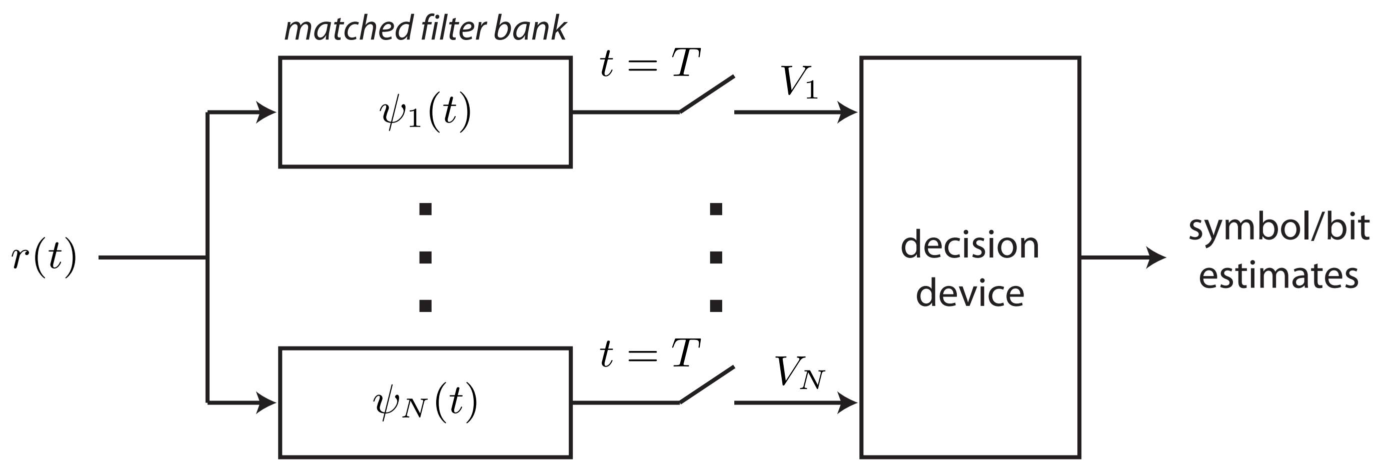
No. We also need to know about how w_n and w_m ($n \neq m$) are related. Since w_n and w_m are jointly Gaussian, we only need to know their covariance.

$$
\begin{aligned}
\mathsf{cov}[w_n] &= \mathsf{E}[w_n w_m] \\
&= \mathsf{E}\left[\left(\int_0^T N(t)\psi_n(t)\,dt\right)\left(\int_0^T N(t)\psi_m(t)\,dt\right)\right] \\
&= \int_0^T \int_0^T \underbrace{\mathsf{E}[N(t_1)N(t_2)]}_{=\frac{N_0}{2}\delta(t_1-t_2)\text{ since white noise}} \psi_n(t_1)\psi_m(t_2)\,dt_1 dt_2 \\
&= \frac{N_0}{2}\int_0^T \psi_n(t)\psi_m(t)\,dt \\
&= 0 \text{ for all } n \neq m \text{ since the basis functions are orthogonal}
\end{aligned}
$$

We now know everything about the noise vector $\boldsymbol{W}$. The JPDF can be written as

$$
f_{\boldsymbol{w}}(\boldsymbol{x}) = \prod_{n=1}^{N} \frac{1}{(\pi N_0)^{1/2}} \exp\left\{\frac{-x_n^2}{N_0}\right\} \text{ (just the product of the marginals)}
$$

Optimal Decisions: Problem Setup



Suppose a symbol was transmitted and we observe the vector $\boldsymbol{v}$ at the output of the sampled matched filter bank. Given the outcome (aka "observation") $\boldsymbol{V} = \boldsymbol{v}$, we want to determine the most likely transmitted message $s_m(t)$, $m \in \{1, \ldots, M\}$. In other words, we will use the following "Maximum Likelihood Decision Rule":

MLDR: *Decide message m was sent if*

$$\mathrm{P}(\text{message } m \text{ sent} \,|\, \boldsymbol{V} = \boldsymbol{v}) \geq \mathrm{P}(\text{message } k \text{ sent} \,|\, \boldsymbol{V} = \boldsymbol{v}) \qquad \forall k \neq m.$$

Analysis - Part 1

We can simplify the decision rule by applying the appropriate form of Bayes' rule:

$$\mathsf{P}(A|X=x) \quad = \quad \mathsf{P}(A)\frac{f_X(x|A)}{f_X(x)}$$

where A is an event, X is a random variable, and x is a particular outcome. We can apply this rule to both sides of our inequality. The MLDR can then be rewritten as

Decide message m was sent if

$$p_m f_{\boldsymbol{V}}(\boldsymbol{v}\,|\,\text{message } m \text{ sent}) \geq p_k f_{\boldsymbol{V}}(\boldsymbol{v}\,|\,\text{message } k \text{ sent}) \qquad \forall k \neq m$$

where p_m is the (a priori) probability that message m was transmitted.

To proceed further, we need an expression for $f_{\boldsymbol{V}}(\boldsymbol{v}\,|\,\text{message } m \text{ sent})$. What do we know?

- Given message m sent, $\boldsymbol{V} = \alpha\boldsymbol{c}_m + \boldsymbol{W}$.
- $\alpha\boldsymbol{c}_m$ is just a constant.
- $\boldsymbol{W}$ is a jointly Gaussian N-dimensional random variable. Each term in $\boldsymbol{W}$ is zero mean with a variance of $N_0/2$ and is independent of all other terms.

Analysis - Part 2

Hence, conditioned on message m transmitted, $\boldsymbol{V}$ is a jointly Gaussian random variable. It isn't too difficult to show that

$$\begin{aligned} E[V_n] &= \alpha c_{m,n} \\ \text{var}[V_n] &= N_0/2 \\ \text{cov}[V_n V_m] &= 0 \text{ if } m \neq n \end{aligned}$$

Hence, the (conditional) joint pdf of V can be written as the product of the (conditional) marginal pdfs

$$\begin{aligned} f_{\boldsymbol{V}}(\boldsymbol{v} \,|\, \text{message } m \text{ sent}) &= \prod_{n=1}^{N} f_{V_n}(v_n \,|\, \text{message } m \text{ sent}) \\ &= \prod_{n=1}^{N} \frac{1}{\sqrt{2\pi N_0/2}} \exp\left[\frac{-(v_n - \alpha c_{m,n})^2}{2N_0/2}\right] \\ &= \frac{1}{(\pi N_0)^{N/2}} \exp\left[-\frac{1}{N_0}\sum_{n=1}^{N}(v_n - \alpha c_{m,n})^2\right] \end{aligned}$$

Analysis - Part 3

Plug this result back into our decision rule (canceling the common terms)...

Decide message m was sent if

$$p_m \exp\left[-\frac{1}{N_0}\sum_{n=1}^{N}(v_n - \alpha c_{m,n})^2\right] \geq p_k \exp\left[-\frac{1}{N_0}\sum_{n=1}^{N}(v_n - \alpha c_{k,n})^2\right] \qquad \forall k \neq m$$

Take natural log of both sides and rearrange...

Decide message m was sent if

$$\sum_{n=1}^{N}(v_n - \alpha c_{m,n})^2 \leq N_0(\ln(p_m) - \ln(p_k)) + \sum_{n=1}^{N}(v_n - \alpha c_{k,n})^2 \qquad \forall k \neq m$$

Define $d(\boldsymbol{x}, \boldsymbol{y}) = \sqrt{(x_1 - y_1)^2 + \cdots + (x_N - y_n)^2}$ as the Euclidean distance measure between the vectors $\boldsymbol{x}$ and $\boldsymbol{y}$. Our decision rule then becomes...

Decide message m was sent if

$$d^2(\boldsymbol{v}, \alpha\boldsymbol{c}_m) \leq d^2(\boldsymbol{v}, \alpha\boldsymbol{c}_k) + N_0 \ln(p_m/p_k) \qquad \forall k \neq m$$

Special Cases and Interpretation

Special case: If all symbols are transmitted with equal probability ($p_m = \frac{1}{M}$ for all m), then $\ln(p_m/p_k) = 0$ and the MLDR can be written as

Decide message m was sent if

$$d(\boldsymbol{v}, \alpha \boldsymbol{c}_m) \leq d(\boldsymbol{v}, \alpha \boldsymbol{c}_k) \qquad \forall k \neq m$$

In other words, given the observation $\boldsymbol{v}$, the most likely transmitted symbol is the one with coordinates nearest to $\boldsymbol{v}$. This is sometimes called the "minimum distance" decision rule.

If the (a priori) transmission probabilities are not all identical, the decision rule is no longer "minimum distance". The term $N_0 \ln(p_m/p_k)$ increases the size of the decision regions for the more likely symbols and decreases the size of the decision regions for the less likely symbols.

Error Probability Analysis - Basic Procedure Pt. 1

1. Draw the geometric representation of the communication system (this may involve Gram-Schmidt orthogonalization). Often, this is given.

2. Determine the optimum decision regions/thresholds. In the case when all symbols are equally likely and the background noise is AWGN, this is just the "minimum distance" rule.

3. (Optional) Take advantage of the fact that the error probability analysis is invariant to rotation and translation of the constellation points. Move the constellation around to simplify things if it makes sense.

4. Write an expression for P(error). An "error" in this case is the event that we decide on a different symbol than the one sent. List all of the ways that an error can occur in terms of the matched filter output vector, i.e.,

$$\mathsf{P}(\text{error}) \quad = \quad \mathsf{P}(\text{symbol 1 sent}, \boldsymbol{V} \in \boldsymbol{\Omega}_1) + \cdots + \mathsf{P}(\text{symbol } M \text{ sent}, \boldsymbol{V} \in \boldsymbol{\Omega}_M)$$

continued...

Error Probability Analysis - Basic Procedure Pt. 2

5. Rewrite the expression for P(error) in terms of the noise vector $\boldsymbol{W}$, i.e.,

$$\mathsf{P}(\text{error}) \;=\; \mathsf{P}(\text{symbol 1 sent}, \boldsymbol{W} \in \boldsymbol{\Lambda}_1) + \cdots + \mathsf{P}(\text{symbol } M \text{ sent}, \boldsymbol{W} \in \boldsymbol{\Lambda}_M)$$

6. Reduce the general P(error) expression as much as possible to Q-functions using the assumptions
   (a) the symbols and the noise are independent
   (b) the symbols are equiprobable (this is our default assumption)
   (c) the noise is Gaussian, independent, and symmetric in each dimension
7. Put the final error probability expression in terms of per-bit SNR, i.e., $\mathcal{E}_{b-av}/N_0$. Make sure you use the original constellation to compute $\mathcal{E}_{b-av}$ and not the rotated/translated constellation.

Error Probability Analysis - Final Comments

- To simplify notation, we will assume $\alpha = 1$ from now on.
- Energy in mth symbol

$$\mathcal{E}_{s_m} = \int_0^T s_m^2(t)\,dt = \int_0^T \left(\sum_{n=1}^{N} c_{m,n}\psi_n(t)\right)^2 dt = \sum_{n=1}^{N}\sum_{k=1}^{N} c_{m,n}c_{m,k}\int_0^T \psi_n(t)\psi_k(t)\,dt$$

but this last integral equals one only if $k = n$. Hence

$$\mathcal{E}_{s_m} = \sum_{n=1}^{N} c_{m,n}^2 = \text{squared magnitude of constellation point.}$$

- Average energy per symbol

$$\mathcal{E}_{s-av} = \sum_{m=1}^{M} p_m \mathcal{E}_{s_m} \underbrace{=}_{\text{if equally likely}} \frac{1}{M}\sum_{m=1}^{M} \mathcal{E}_{s_m}$$

- Average energy per bit

$$\mathcal{E}_{b-av} = \frac{\mathcal{E}_{s-av}}{\log_2(M)}$$

4-PAM Probability of Symbol Error Analysis

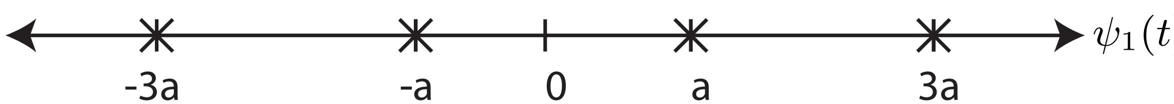
Suppose

$$s(t) = \begin{cases} \frac{1}{\sqrt{T}} & 0 \leq t \leq T \\ 0 & \text{otherwise} \end{cases}$$

and our symbols are given as

$$\begin{aligned} s_1(t) &= 3as(t) \\ s_2(t) &= as(t) \\ s_3(t) &= -as(t) \\ s_t(t) &= -3as(t) \end{aligned}$$

Step 1 is pretty easy: $\psi_1(t) = s(t)$.



Assuming the symbols are all transmitted with equal probability, we can easily draw the decision regions in this one-dimensional system. Rest of analysis on blackboard...

M-PAM Symbol Error Probability Analysis

We now know that, for a 4-PAM communication system, the probability of an erroneous symbol decision can be expressed as

$$P(\text{error}) \quad = \quad \frac{3}{2} Q\left(\sqrt{\frac{4\mathcal{E}_{b-av}}{5N_0}}\right)$$

The same analysis can be repeated to show that, for general M-PAM, the probability of symbol error can be expressed as

$$P(\text{error}) \quad = \quad \frac{2(M-1)}{M} Q\left(\sqrt{\frac{6\log_2(M)\mathcal{E}_{b-av}}{(M^2-1)N_0}}\right)$$

General Remarks

1. The quantity $\frac{\mathcal{E}_{b-av}}{N_0}$ is commonly called the "SNR-per-bit" or the "per-bit-SNR". This is the *only* fair way to compare the error probability of two communication systems as a function of the transmit energy.
2. The probability of symbol error is commonly called the "symbol error rate" and is abbreviated as SER.
3. How does M affect the communication system?
   (a) Increasing M means, in general, that you transmit more bits per symbol (good) but also means, in general, that the SER will increase if the SNR-per-bit is held fixed (bad).
   (b) For a given modulation format, e.g. PAM, the bandwidth required is roughly inversely proportional to $\log_2(M)$. For example, an 8-PAM communication system requires about $\frac{1}{3}$ the bandwidth of a 2-PAM communication system with the same overall bit rate.

This is a <u>fundamental tradeoff</u> in designing communication systems: *bandwidth efficiency* (ratio of data rate in bits/sec to required bandwidth in Hz) versus *SER* versus *SNR-per-bit*.

Tradeoff Example

Suppose we wanted to transmit 6000 bits/sec at $\frac{\mathcal{E}_{b-av}}{N_0}$ =10dB. We can use 2-PAM, 4-PAM, or 8-PAM. What are the tradeoffs?

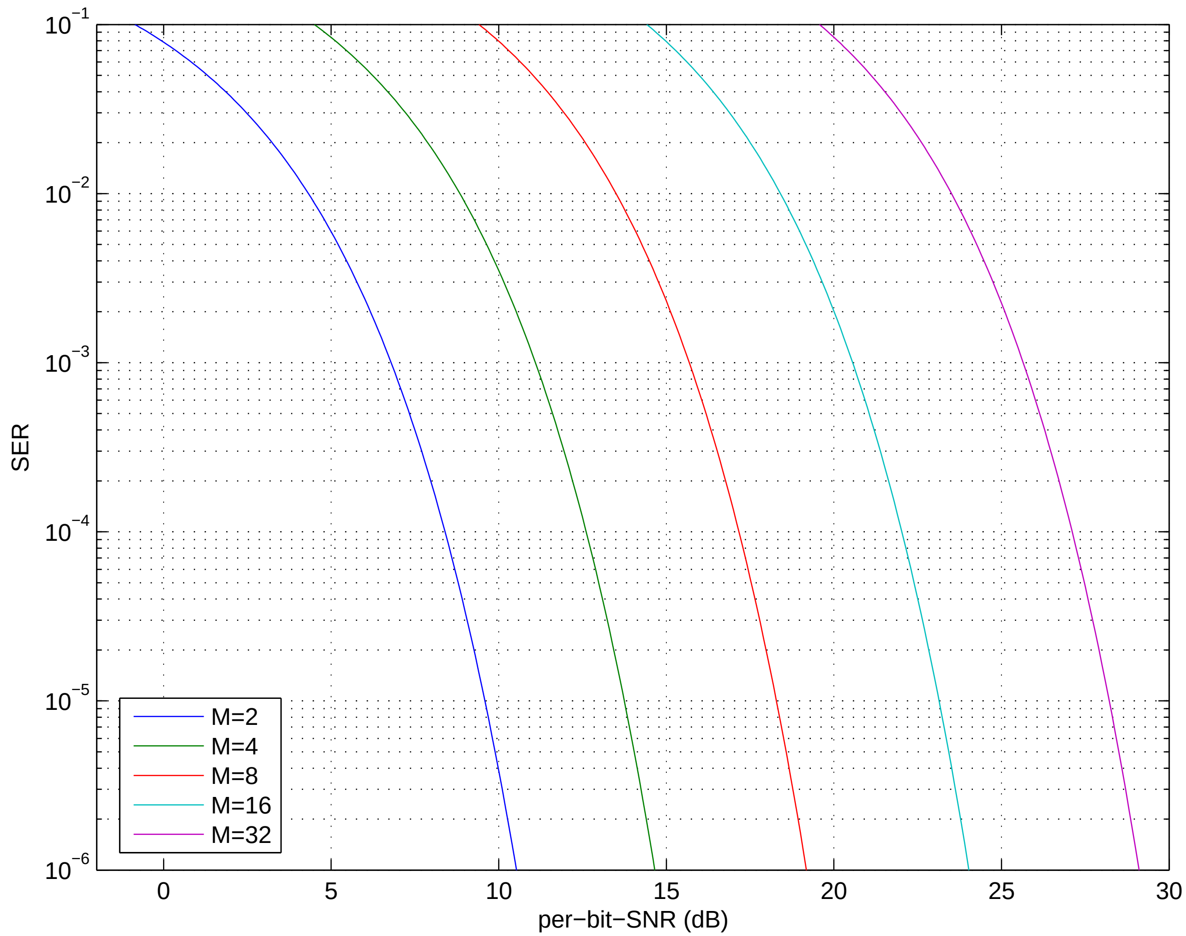
	2-PAM	4-PAM	8-PAM
required bandwidth	$2B$	B	$\frac{2B}{3}$
SER	5×10^{-6}	3×10^{-3}	9×10^{-2}
$\frac{\mathcal{E}_{b-av}}{N_0}$ required to have SER$=5 \times 10^{-6}$	10dB	14.5dB	19dB

8-PAM requires 1/3 the bandwidth but requires 9dB more SNR-per-bit to get the same symbol error rate.

Bottom line: The "optimum" choice depends on your application.

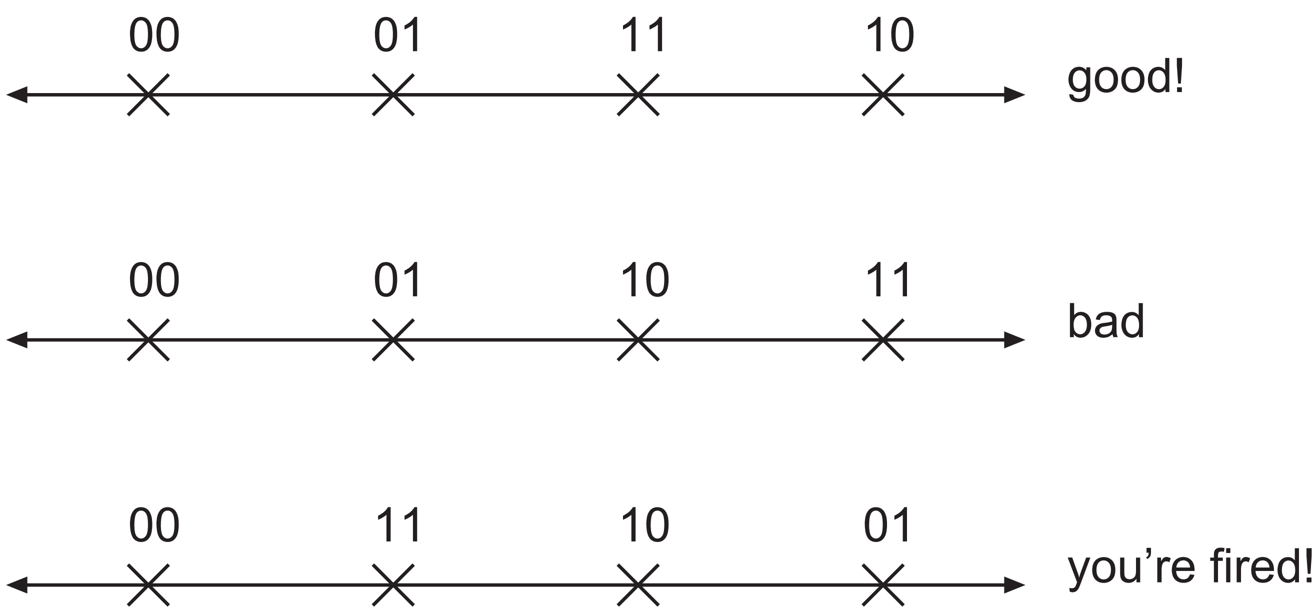
Plotting SER versus Per-Bit-SNR

Almost all SER vs. per-bit-SNR plots show SER in log scale versus $\frac{\mathcal{E}_{b-av}}{N_0}$ in dB.



Assigning Bits to Symbols

Basic idea: Some symbol errors are more common than others. We want the most common symbol errors to cause only one bit error.



Can use "Gray code" for one-dimensional constellations. Higher dimensional constellations are more difficult. In many cases it may not be possible to have neighboring constellation points differ by only one bit.

Bit-Error-Rate Versus Symbol-Error-Rate

When $M = 2$, BER=SER. For $M > 2$, an exact relationship is difficult to derive (and depends on how you assigned the bits to the symbols).

Since each symbol contains $M = \log_2(M)$ bits, a symbol error results in at least one bit error and at most $\log_2(M)$ bit errors. Hence, the BER is bounded by

$$\frac{\text{SER}}{\log_2(M)} \leq \text{BER} \leq \text{SER}.$$

If the bits are assigned to symbols in a "smart" way, then most symbol errors result in only one bit error and

$$\text{BER} \approx \frac{\text{SER}}{\log_2(M)}.$$