ECE4304 D-term 2007: Lecture 29 Supplemental Slides

D. Richard Brown III
Worcester Polytechnic Institute, Department of Electrical and Computer Engineering

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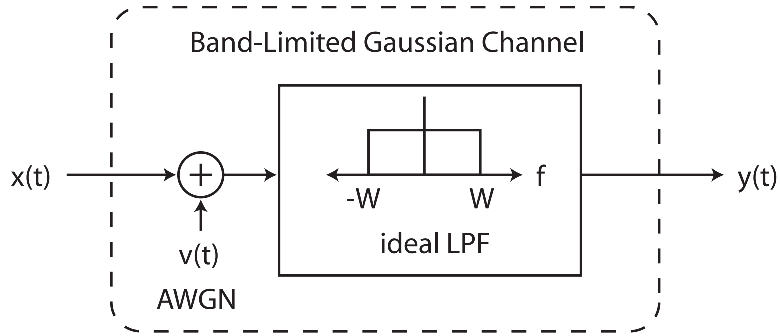
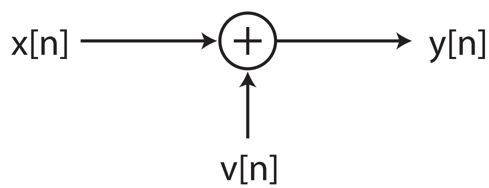
Capacity of the Band-Limited Gaussian Channel

Suppose we have a communication system with a bandlimited channel and additive white Gaussian noise with PSD $\frac{N_0}{2}$ as shown below.



Since everything is bandlimited, we can sample the signals at the Nyquist rate $T = \frac{1}{2W}$ to get the equivalent discrete-time system shown below.



Note that, unlike our previous analysis of DMCs, $x[n]$, $y[n]$, and $v[n]$ are *continuous valued* here. The noise $v[n]$ is zero mean, Gaussian, with variance $\sigma_V^2 =$

Capacity of the Band-Limited Gaussian Channel

We again consider codewords (transmit sequences) of length N.

$$\begin{aligned}
\underline{\mathsf{X}} &= [x[1], \ldots, x[N]] \\
\underline{\mathsf{V}} &= [v[1], \ldots, v[N]] \\
\underline{\mathsf{Y}} &= [y[1], \ldots, y[N]] = [x[1], \ldots, x[N]] + [v[1], \ldots, v[N]]
\end{aligned}$$

To have a meaningful measure for capacity, we must apply a power constraint to the the transmitted sequences, i.e.,

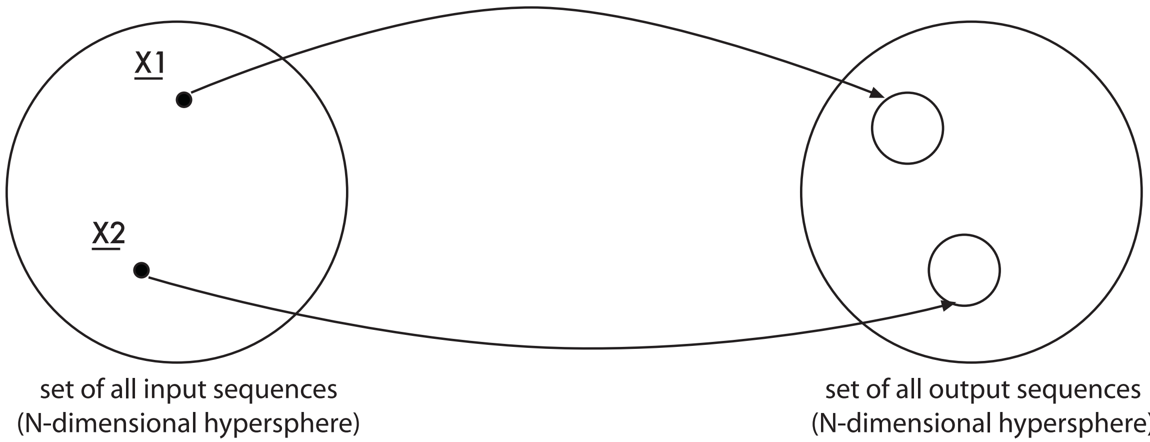
$$\frac{1}{N} \sum_{n=1}^{N} x^2[n] \leq P_x$$

where P_x is the maximum permitted (average) power of the source.

To determine the capacity of this channel, we can use our understanding of "typical sequences" again. The number of different sequences that we can we send through this channel with arbitrarily small probability of sequence error is related to the size of the set of typical sequences at the output of the channel.

Capacity of the Band-Limited Gaussian Channel

Assuming N is large, the set of typical output sequences conditioned on a particular $\underline{\mathsf{X}}$ is much smaller than the set of unconditional typical output sequences.



Let K denote the number of sequences that can be sent reliably. K is just the ratio of the volumes:

$$K = \frac{\text{volume of big hypersphere}}{\text{volume of little hypersphere}}$$

The volume of an N-dimensional hypersphere of radius r is given as volume $= \alpha_N r^N$ where α_N is a constant that only depends on the dimension N (and not the radius).

Capacity of the Band-Limited Gaussian Channel

Q: What is the radius of the little spheres?

Intuitively, if the noise power is small, these spheres should be small. The Euclidean distance between the output sequence $\underline{\mathsf{Y}}$ and the input sequence $\underline{\mathsf{X}}$ is given as

$$\begin{aligned} d(\underline{\mathsf{X}}, \underline{\mathsf{Y}}) &= \sqrt{(x[1]-y[1])^2 + \cdots + (x[N]-y[N])^2} \\ &= \sqrt{v^2[1] + \cdots + v^2[N]}. \end{aligned}$$

Note that $d(\underline{\mathsf{X}}, \underline{\mathsf{Y}})$ is a random variable but, as $N \to \infty$, the law of large numbers tells us that

$$d(\underline{\mathsf{X}}, \underline{\mathsf{Y}}) \approx \sqrt{NWN_0}$$

with high probability.

This implies that, given a particular input sequence $\underline{\mathsf{X}}$ was sent, the output sequence $\underline{\mathsf{Y}}$ is highly likely to be somewhere in a "thin spherical shell" centered around $\underline{\mathsf{X}}$ of radius approximately $\sqrt{NWN_0}$. We don't know where we will land in this shell, we only know that we land somewhere in it with very high probability.

Capacity of the Band-Limited Gaussian Channel

Q: What is the radius of the big sphere?

Intuitively, the big sphere gets bigger if the noise power and/or the signal power increases.

$$
\begin{aligned}
\text{radius} = d(\underline{\mathsf{Y}}, \underline{0}) &= \sqrt{y^2[1] + \cdots + y^2[N]} \\
&= \sqrt{(x[1] + v[1])^2 + \cdots + (x[N] + v[N])^2}.
\end{aligned}
$$

Again, $d(\underline{\mathsf{Y}}, \underline{0})$ is a random variable. The sphere is largest when we apply the maximum input power. In this case, the law of large numbers tells us that

$$
d(\underline{\mathsf{Y}}, \underline{0}) \approx \sqrt{N(P_x + WN_0)}
$$

with high probability where we have assumed that $\underline{\mathsf{X}}$ and $\underline{\mathsf{V}}$ are independent.

Note that, since we control the input power P_x, we are not constrained to $\underline{\mathsf{Y}}$ always being in a "thin spherical shell" of this radius.

Capacity of the Band-Limited Gaussian Channel

Taking the ratio of the volumes, we get

$$K = \frac{(N(P_x + WN_0))^{N/2}}{(NWN_0)^{N/2}}$$

distinct sequences (aka codewords) that can be sent reliably through the band-limited Gaussian channel. This implies that we can convey $\log_2(K)$ bits of information per codeword and have arbitrarily small probability of information error.

The capacity (bits of information per symbol) of the channel is then

$$C = \frac{1}{N}\log_2 K = \frac{1}{2}\log_2\left(\frac{P_x}{WN_0} + 1\right) \text{ bits of information per symbol.}$$

Since $T = \frac{1}{2W}$, the implied symbol rate of this system is $2W$ symbols per second. This implies that the capacity of the bandlimited Gaussian channel can be written as

$$C = W\log_2\left(\frac{P_x}{WN_0} + 1\right) \text{ bits of information per second.}$$

This is probably the most famous result from Shannon's work in 1948. The quantity $\frac{P_x}{WN_0}$ is commonly called the SNR of the channel.

Examples and Remarks

Good telephone channel: W=3.6kHz, SNR=30dB. What is the capacity?

$$C = 3600\log_2(1000+1) = 35882 \text{ bits of information per second.}$$

Bad telephone channel: W=3.6kHz, SNR=10dB. What is the capacity?

$$C = 3600\log_2(10+1) = 12454 \text{ bits of information per second.}$$

What happens when we double the bandwidth of the "good" telephone channel?

- W=7200Hz
- SNR=27dB (why?)

In this case

$$C = 7200\log_2(500+1) = 64574 \text{ bits of information per second.}$$

Doubling the bandwidth increases capacity but does not double the capacity due to the increased noise power in Y.