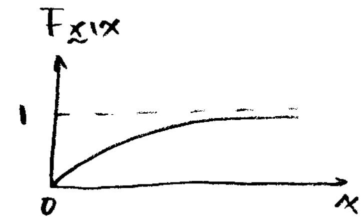
# ECE 4304 HW # 1

1. a) $\int_0^\infty f_{\underline{X}}(x) = 1 \implies \int_0^\infty c\exp(-\alpha x)\,dx = 1$

$\implies c = \alpha$

b) $F_{\underline{X}}(x) = \int_0^x f_X(x)\,dx = \int_0^x \alpha\exp(-\alpha x)\,dx$

$= 1 - \exp(-\alpha x) \qquad 0 \le x < \infty$




when $\alpha = 1$, the plot is shown on right.

c) $E[X] = \int_0^\infty x f_X(x)\,dx = \int_0^\infty x\,\alpha\exp(-\alpha x)\,dx$

$= \frac{1}{a}$

d) $Var[X] = \int_0^\infty (x - \frac{1}{a})^2 f_X(x)\,dx$

$= \frac{1}{a^2}$

2. $P_e = P(0 \text{ sent})\,P(\text{got } 1 \mid 0 \text{ sent})$
$+ P(1 \text{ sent})\,P(\text{got } 0 \mid 1 \text{ sent})$

$= P(V=-1)\,P(X>0 \mid V=-1) + P(V=1)\,P(X<0 \mid V=1)$

$= 0.5\left[P(w-1>0) + P(1+w<0)\right]$

$= 0.5\,(4.2906e{-}004 + 4.2906e{-}004)$

$= 4.2906\times10^{-4}$

3. a)

$$\int_{-1}^{1}\int_{-1}^{1} f_{XY}(x,y)\,dx\,dy = 1$$

$$\Rightarrow \int_{-1}^{1}\int_{-1}^{1} c(x-y)^2\,dx\,dy = 1$$

$$\Rightarrow \quad c = \frac{3}{8}$$

b)

$$F_{XY}(x,y) = \int_{-1}^{x}\int_{-1}^{y} f_{uv}(u,v)\,du\,dv$$

$$= \frac{1}{16}(x+1)(y+1)(2x^2-3xy+x+2y^2+y+1)$$

c)

$$f_X(x) = \int_{-1}^{1} f_{XY}(x,y)\,dy$$

$$= \frac{1}{4}+\frac{3}{4}x^2$$

$$f_Y(x) = \int_{-1}^{1} f_{XY}(x,y)\,dx$$

$$= \frac{1}{4}+\frac{3}{4}y^2$$

d)

$$P[X>0, Y>0] = \int_0^1\int_0^1 f_{XY}(x,y)\,dx\,dy$$

$$= \frac{1}{16}$$

e)

$$\mathrm{corr}[XY] = E[XY]$$

$$= \int_{-1}^{1}\int_{-1}^{1} XY f_{XY}(x,Y)\,dx\,dy$$

$$= -\frac{1}{3}$$

(f) $\mu_X = \int_{-1}^{1} x f_X(x)\,dx = 0$

the same, $\mu_Y = 0$

$$\text{cov}[XY] = E[XY] - \mu_X \mu_Y$$
$$= -\frac{1}{3}$$

4) a) $Z(t_1)$ and $Z(t_2)$ are Gaussian RVs, so their joint probability density function is of the form

$$f(Z(t_1), Z(t_2)) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left\{-\frac{1}{2(1-\rho^2)}\left(\frac{(Z_1-\eta_1)^2}{\sigma_1^2} - 2\rho\frac{(Z_1-\eta_1)(Z_2-\eta_2)}{\sigma_1\sigma_2} + \frac{(Z_2-\eta_2)^2}{\sigma_2^2}\right)\right\}$$

$\rho$ is the correlation coefficient

$$\left.\begin{array}{l} \eta_1 = E[Z(t_1)] = \cos(2\pi t_1)E[X] + \sin(2\pi t_1)E[Y] \\ E[X] = E[Y] = 0 \end{array}\right\} \Rightarrow E[Z(t_1)] = 0$$

Similarly $E[z(t_2)]=0$

So $Cov[z(t_1)z(t_2)] = E[z(t_1)z(t_2)]$

$= \cos(2\lambda t_1)\cos(2\lambda t_2)E[X^2]$

$+ [\cos(2\lambda t_1)\sin(2\lambda t_2) + \sin(2\lambda t_1)\cos(2\lambda t_2)]\, E[XY]$

$+ \sin(2\lambda t_1)\sin(2\lambda t_2)\, E[Y^2]$

where $E[XY]=0$ ( $X, Y$ are independent and $E[X]=E[Y]=0$)

$E[X^2] = \sigma_X^2 + (E[X])^2 = 1$

similarly $E[Y^2]=1$

Hence $Cov[z(t_1)z(t_2)] = \cos[2\lambda(t_1-t_2)]$

when $t_1=t_2$ covariance turns into variance

so $\sigma^2_{z(t_1)} = \sigma^2_{z}(t_2) = 1$

Therefore, $\rho = \dfrac{Cov[z(t_1)z(t_2)]}{\sigma_{z(t_1)}\sigma_{z(t_2)}}$

$= \cos[2\lambda(t_1-t_2)]$

Hence the joint probability density function is

$f_{z(t_1)z(t_2)}(z_1, z_2) =$

$$\frac{1}{2\lambda\sin[2\lambda(t_1-t_2)]} \exp\left\{ \frac{1}{2\sin^2[2\lambda(t_1-t_2)]}\left(z_1^2 - 2\cos[2\lambda(t_1-t_2)]z_1z_2 + z_2^2\right) \right\}$$

b) The covariance of $Z(t_1)$ and $Z(t_2)$ only depends on the time difference, So $Z(t)$ is wide sense stationary. Since it is Gaussian it is also strictly stationary.

5 a) by observation,

$$F_{\underset{\sim}{X}(t)}(x) = \begin{cases} 0, & x < 0 \\ \frac{1}{2}, & 0 \le x \le A \\ 1, & x > A \end{cases}$$

correspondly,

$$f_{\underset{\sim}{X}(t)}(X) = \frac{1}{2}\delta(x) + \frac{1}{2}\delta(x-A)$$

b) By ensemble-averaging,

$$E[X(t)] = \int_{-\infty}^{\infty} x f_{X(t)}(x)\,dx$$

$$= \frac{A}{2}$$

$$R_{\underset{\sim}{X}}(z) = E[X(t+z)X(t)]$$
$$= \int_{-\infty}^{\infty} X(t)X(t+z) f_{T_d}(t_d)\,dt_d$$

Consider only one period:

$$X(t) = \begin{cases} A & t - \frac{T_0}{2} < t_d < t \\ 0 & t < t_d < \frac{T_0}{2} + t \end{cases}$$

$$X(t+z) = \begin{cases} A & t - \frac{T_0}{2} + z < t_d < t + z \\ 0 & t + z < t_d < \frac{T_0}{2} + t + z \end{cases}$$

Hence $R_{\underset{\sim}{X}}(\tau) = \begin{cases} A^2 \int_{t-\frac{T_0}{2}+\tau}^{t} \frac{1}{T_0} \, dt_d & \tau > 0 \\ A^2 \int_{t-\frac{T_0}{2}}^{t+\tau} \frac{1}{T_0} \, dt_d & \tau < 0 \end{cases}$

$$= \frac{A^2}{2}\left(1 - \frac{2|\tau|}{T_0}\right) \qquad |\tau| \leq \frac{T_0}{2}$$

$R_{\underset{\sim}{X}}(\tau)$ is periodic with period $T_0$.

c) By time-averaging,

By observation,

$$\mu_x = \frac{A}{2}$$

$$R_x(\tau, T_0) = \frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} X(t+\tau) X(t) \, dt$$

$$= \frac{A^2}{2}\left(1 - 2\frac{|\tau|}{T_0}\right), \quad |\tau| \leq \frac{T_0}{2}$$

$R_x(\tau)$ is periodic with period $T_0$

d) For the reason that ensemble-averaging and time-averaging yield the same results for the mean and autocorrelation, $X(t)$ is ergodic in both mean and autocorrelation function. The ergodicity also implies that $X(t)$ is wide sense stationary.

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%-----------------------------------------------------------------------------
%ECE4304 Hw#1
%problem 6
%-----------------------------------------------------------------------------
N=1e6;                      % # of iterations
W=0.3*randn(N,1);           %Gaussion Noise with mean=0, standard deviation=0.3
temp=rand(N,1);
V=(temp>0.5)-(temp<0.5);    %transmite signal
X=V+W;                      %corrupted signal
Y=sign(X);                  %reciever decision
Pe=1-sum(Y==V)/N            %error probability
```