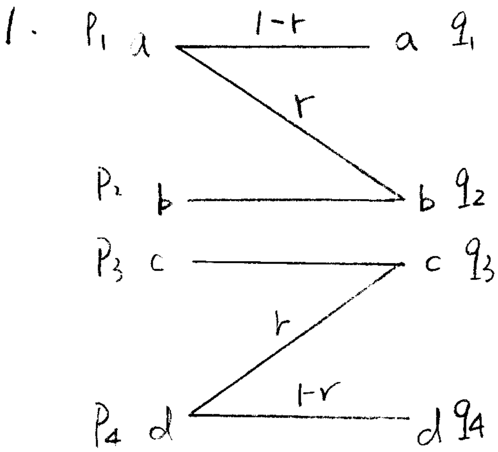
# HW # 6

1.




$P_a = P_d$

$P_b = P_c$

$P_a + P_b = 0.5$

$P_c + P_d = 0.5$

$\Rightarrow$ $P_4 = P_1$, $P_3 = P_2$, $P_2 = 0.5 - P_1$

$$I = H(Y) - H(Y|X)$$

$$= -\sum_{k=1}^{4} q_k \log_2(q_k) + \sum_{m=1}^{4}\sum_{k=1}^{k} P_{m,k}\log_2(P_{k|m})$$

| | | |
|---|---|---|
| $q_1 = P_1(1-r)$ | $P_{1\|1} = 1-r$ | $P_{3\|3} = 1$ |
| $q_2 = P_1 r + P_2$ | $P_{2\|1} = r$ | $P_{4\|3} = 0$ |
| $q_3 = P_3 + P_4 r$ | $P_{1\|2} = 0$ | $P_{3\|4} = r$ |
| $q_4 = P_4(1-r)$ | $P_{2\|2} = 1$ | $P_{4\|4} = 1-r$ |

| | |
|---|---|
| $P_{1,1} = P_1(1-r)$ | $P_{3,3} = P_3$ |
| $P_{2,1} = P_1 r$ | $P_{4,3} = 0$ |
| $P_{1,2} = 0$ | $P_{3,4} = P_4 r$ |
| $P_{2,2} = P_2$ | $P_{4,4} = P_4(1-r)$ |

$$I = H(Y) - H(Y|X) = -2P_1(1-r)\log_2(P_1) - 2(P_1 r + \tfrac{1}{2} - P_1)\log_2(P_1 r + \tfrac{1}{2} - P_1) + 2P_1 r\log_2 r$$

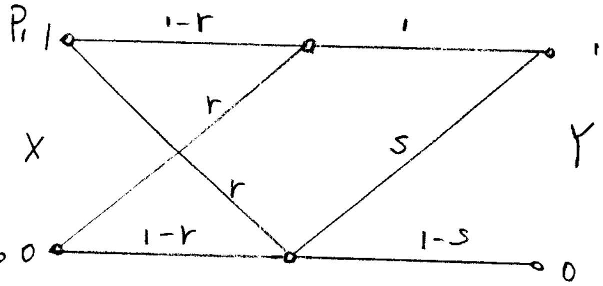
$$P_1 = \underset{0<P_1<\frac{1}{2}}{\arg\max}\, I \Rightarrow \boxed{P_1 = \frac{1}{2r^{\frac{r}{r-1}} - 2(r-1)}}$$

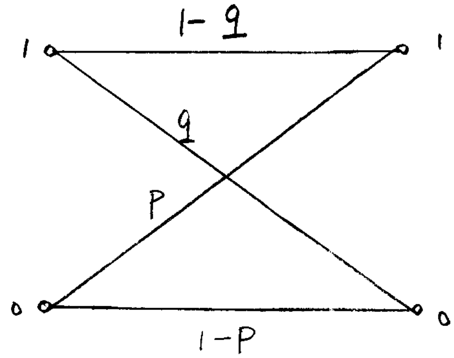
$$\Rightarrow C = \{-\log_2[\tfrac{1}{2}(1+r^{\frac{r}{1-r}}-r)]^{1-r} + r^{\frac{r}{1-r}}\log_2\left(\frac{r^{(\frac{r}{1-r})}}{2(1+r^{\frac{r}{1-r}}-r)}\right) - r\log_2 r\}/(1-r^{\frac{r}{1-r}}+r-1)$$

when $r=0$ $P_1 = \frac{1}{4}$ $C = 2$

$r = 1$ $\lim_{r\to 1} C = 1$

2.




$$P(y_1|x_1) = 1-r+rs = 1-q$$
$$P(y_1|x_0) = r+(1-r)s = p$$
$$P(y_0|x_1) = r(1-s) = q$$
$$P(y_0|x_0) = (1-r)(1-s) = 1-p$$

$$P(y_1) = P_1(1-q)+P_0 p$$
$$P(y_0) = P_0(1-p)+P_1 q$$

$$I = H(Y) - H(Y|X)$$
$$= -(P_1(1-q)+P_0 p)\log_2(P_1(1-q)+P_0 p)$$
$$-(P_0(1-p)+P_1 q)\log_2(P_0(1-p)+P_1 q)$$
$$+P_1(1-q)\log_2(1-q)+P_0 p\log_2 p+P_1 q\log_2 q+P_0(1-p)\log_2(1-p)$$

$$P_1+P_0=1$$

$$\Rightarrow P_1 = \underset{0\le P_1\le 1}{\arg\max}\, I = \frac{(1-q)^{\frac{q-1}{q+p-1}}(1-p)^{\frac{q}{q+p-1}}p^{\frac{p}{q+p-1}}q^{\frac{-q}{q+p-1}}-p}{\left(1+(1-q)^{\frac{q-1}{q+p-1}}(1-p)^{\frac{1-p}{q+p-1}}p^{\frac{p}{q+p-1}}q^{\frac{-q}{q+p-1}}\right)(1-q-p)}$$

3. a) $C = B \log_2 (1 + \frac{P}{N_0 B})$ $SNR = 10^2 = 20dB$

$= B \log_2 (1 + SNR)$

$= 3.4 \times 10^3 \log_2 (1 + 10^2)$

$= 2.2638 \times 10^4$ bits/s

b) let $\vec{X}$ denote the symbols, the symbols are equiprobable

then $H(\vec{X}) = -\log_2 \frac{1}{128}$

$= 7$ bits/symbol

so $r_{symbol} = \frac{C}{7} = 3.234 \times 10^3$ symbol/s

4. $\frac{\mathcal{E}_b}{N_0} = 3\,dB = 10^{0.3}$

(7,4) Hamming coded system $\quad t=1$

$$P_e = Q\left(\sqrt{\frac{2\frac{4}{7}\mathcal{E}_b}{N_0}}\right) = 0.0655$$

P( uncorrectable codeword error )

$$= 1-(1-P_e)^7 - 7P_e(1-P_e)^6$$

$$= 0.0723$$

$$IER = P(\text{u.c.w.e})/4$$

$$= 0.0181$$

(15,11) Hamming coded system $\quad t=1$

$$P_e = Q\left(\sqrt{\frac{2\frac{11}{15}\mathcal{E}_b}{N_0}}\right) = 0.0436$$

$$P(\text{u.c.w.e}) = 1-(1-P_e)^{15} - 15P_e(1-P_e)^{14}$$

$$= 0.1372$$

$$IER = P(\text{u.c.w.e})/11$$

$$= 0.0125$$

(15,7) BCH coded system $t=2$

$$P_e = Q\left(\sqrt{\frac{2\frac{7}{15}\mathcal{E}_b}{N_0}}\right) = 0.0862$$

$$P(\text{u.c.w.e}) = 1-(1-P_e)^{15} - 15P_e(1-P_e)^{14} - 105P_e^2(1-P_e)^{13}$$

$$= 0.1336$$

$$IER = P(\text{u.c.w.R})/7$$

$$= 0.0191$$

(15,5) BCH coded system $t=3$

$$P_e = Q\left(\sqrt{\frac{2\frac{5}{15}\mathcal{E}_b}{N_0}}\right) = 0.1244$$

$$P(\text{u.c.w.e}) = 1-(1-P_e)^{15} - 15P_e(1-P_e)^{14} - 105P_e^2(1-P_e)^{13}$$

$$- 455P_e^3(1-P_e)^{12}$$

$$= 0.1063$$

$$IER = P(\text{u.c.w.e})/5$$

$$= 0.0213$$

Uncoded system

$$P_e = Q\left(\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right) = 0.0229 = IER$$