ECE4703: Lecture 9

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FFT Complexity Analysis: $N = 1$

When $N = 1$, there is nothing to divide into even and odd parts, so we will just use the DFT. The DFT equation:

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j2\pi kn/N} \qquad \text{for } k = 0, 1, \ldots, N-1$$

When $N = 1$, we have

$$X[0] = x[0]e^{0} = x[0]$$

No multiplies or additions. Hence a one-point DFT/FFT has no MACs.

FFT Complexity Analysis: $N = 2$

Two-point FFT computed using radix-2 decimation in time:

$$\begin{aligned} X[0] &= X_{even}[0] + e^{-j2\pi \cdot 0/2} X_{odd}[0] = X_{even}[0] + X_{odd}[0] \\ X[1] &= X_{even}[1] + e^{-j2\pi \cdot 1/2} X_{odd}[1] = X_{even}[1] - X_{odd}[1] \end{aligned}$$

Note that $X_{even}[k]$ and $X_{odd}[k]$ for $k = 0, 1$ are just one-point FFTs.

$$\begin{aligned} X_{even}[0] &= \mathsf{DFT}_1\{x[0], 0\} = x[0] \\ X_{odd}[0] &= \mathsf{DFT}_1\{x[1], 0\} = x[1] \\ X_{even}[1] &= \mathsf{DFT}_1\{x[0], 1\} = x[0] \\ X_{odd}[1] &= \mathsf{DFT}_1\{x[1], 1\} = x[1] \end{aligned}$$

Hence,

$$\begin{aligned} X[0] &= x[0] + x[1] \\ X[1] &= x[0] - x[1] \end{aligned}$$

How many MACs in the R2-DIT two-point FFT?

Note that the two-point DFT and FFT are identical. Still no gain.

An Observation

Look again at the one-point FFTs:

$$\begin{aligned} X_{even}[0] &= \mathsf{DFT}_1\{x[0], 0\} = x[0] \\ X_{odd}[0] &= \mathsf{DFT}_1\{x[1], 0\} = x[1] \\ X_{even}[1] &= \mathsf{DFT}_1\{x[0], 1\} = x[0] \\ X_{odd}[1] &= \mathsf{DFT}_1\{x[1], 1\} = x[1] \end{aligned}$$

Note that $X_{even}[1] = X_{even}[0]$ and $X_{odd}[1] = X_{odd}[0]$.

This is a consequence of the periodicity of the DFT/FFT. Recall that, in general, a P-point DFT/FFT satisfies

$$X[k + \ell P] = \mathsf{DFT}_P\{\{x\}, k + \ell P\} = X[k]$$

This property is critical to getting the desired complexity reduction.

FFT Complexity Analysis: $N = 4$ (part 1)

Four-point FFT computed using radix-2 decimation in time:

$$\begin{aligned}
X[0] &= X_{even}[0] + e^{-j2\pi\cdot 0/4}X_{odd}[0] = X_{even}[0] + X_{odd}[0] \\
X[1] &= X_{even}[1] + e^{-j2\pi\cdot 1/4}X_{odd}[1] = X_{even}[1] - jX_{odd}[1] \\
X[2] &= X_{even}[2] + e^{-j2\pi\cdot 2/4}X_{odd}[2] = X_{even}[2] - X_{odd}[2] \\
X[3] &= X_{even}[3] + e^{-j2\pi\cdot 3/4}X_{odd}[3] = X_{even}[3] + jX_{odd}[3]
\end{aligned}$$

Remarks:

- $X_{even}[k]$ and $X_{odd}[k]$ for $k = 0, 1, 2, 3$ are all results from two-point FFTs.
- Recall that the two-point DFT/FFT $\{X[0], X[1]\} = \text{FFT}_2\{x[0], x[1]\}$ requires two MACs to return two values.
- The periodicity of the DFT/FFT implies that

$$\begin{aligned}
X_{even}[2] &= X_{even}[0] \\
X_{even}[3] &= X_{even}[1] \\
X_{odd}[2] &= X_{odd}[0] \\
X_{odd}[3] &= X_{odd}[1]
\end{aligned}$$

Hence, we only need to compute: $\{X_{even}[0], X_{even}[1]\}$, $\{X_{odd}[0], X_{odd}[1]\}$.

FFT Complexity Analysis: $N = 4$ (part 2)

Four-point FFT computed using radix-2 decimation in time:

$$\begin{aligned}
X[0] &= X_{even}[0] + e^{-j2\pi \cdot 0/4} X_{odd}[0] = X_{even}[0] + X_{odd}[0] \\
X[1] &= X_{even}[1] + e^{-j2\pi \cdot 1/4} X_{odd}[1] = X_{even}[1] - jX_{odd}[1] \\
X[2] &= X_{even}[2] + e^{-j2\pi \cdot 2/4} X_{odd}[2] = X_{even}[2] - X_{odd}[2] \\
X[3] &= X_{even}[3] + e^{-j2\pi \cdot 3/4} X_{odd}[3] = X_{even}[3] + jX_{odd}[3]
\end{aligned}$$

Computation of $\{X_{even}[0], X_{even}[1]\} = \mathsf{FFT}_2\{x[0], x[2]\}$ requires how many MACs?

FFT Complexity Analysis: $N = 4$ (part 2)

Four-point FFT computed using radix-2 decimation in time:

$$\begin{aligned}
X[0] &= X_{even}[0] + e^{-j2\pi\cdot 0/4} X_{odd}[0] = X_{even}[0] + X_{odd}[0] \\
X[1] &= X_{even}[1] + e^{-j2\pi\cdot 1/4} X_{odd}[1] = X_{even}[1] - jX_{odd}[1] \\
X[2] &= X_{even}[2] + e^{-j2\pi\cdot 2/4} X_{odd}[2] = X_{even}[2] - X_{odd}[2] \\
X[3] &= X_{even}[3] + e^{-j2\pi\cdot 3/4} X_{odd}[3] = X_{even}[3] + jX_{odd}[3]
\end{aligned}$$

Computation of $\{X_{even}[0], X_{even}[1]\} = \mathsf{FFT}_2\{x[0], x[2]\}$ requires how many MACs? 2

Computation of $\{X_{odd}[0], X_{odd}[1]\} = \mathsf{FFT}_2\{x[1], x[3]\}$ requires how many MACs?

FFT Complexity Analysis: $N = 4$ (part 2)

Four-point FFT computed using radix-2 decimation in time:

$$\begin{aligned}
X[0] &= X_{even}[0] + e^{-j2\pi \cdot 0/4} X_{odd}[0] = X_{even}[0] + X_{odd}[0] \\
X[1] &= X_{even}[1] + e^{-j2\pi \cdot 1/4} X_{odd}[1] = X_{even}[1] - jX_{odd}[1] \\
X[2] &= X_{even}[2] + e^{-j2\pi \cdot 2/4} X_{odd}[2] = X_{even}[2] - X_{odd}[2] \\
X[3] &= X_{even}[3] + e^{-j2\pi \cdot 3/4} X_{odd}[3] = X_{even}[3] + jX_{odd}[3]
\end{aligned}$$

Computation of $\{X_{even}[0], X_{even}[1]\} = \text{FFT}_2\{x[0], x[2]\}$ requires how many MACs? 2

Computation of $\{X_{odd}[0], X_{odd}[1]\} = \text{FFT}_2\{x[1], x[3]\}$ requires how many MACs? 2

How many more MACs are required to assemble the results into a four-point FFT?

FFT Complexity Analysis: $N = 4$ (part 2)

Four-point FFT computed using radix-2 decimation in time:

$$\begin{aligned}
X[0] &= X_{even}[0] + e^{-j2\pi\cdot 0/4}X_{odd}[0] = X_{even}[0] + X_{odd}[0] \\
X[1] &= X_{even}[1] + e^{-j2\pi\cdot 1/4}X_{odd}[1] = X_{even}[1] - jX_{odd}[1] \\
X[2] &= X_{even}[2] + e^{-j2\pi\cdot 2/4}X_{odd}[2] = X_{even}[2] - X_{odd}[2] \\
X[3] &= X_{even}[3] + e^{-j2\pi\cdot 3/4}X_{odd}[3] = X_{even}[3] + jX_{odd}[3]
\end{aligned}$$

Computation of $\{X_{even}[0], X_{even}[1]\} = \text{FFT}_2\{x[0], x[2]\}$ requires how many MACs? 2

Computation of $\{X_{odd}[0], X_{odd}[1]\} = \text{FFT}_2\{x[1], x[3]\}$ requires how many MACs? 2

How many more MACs are required to assemble the results into a four-point FFT? 4

Hence, the total MACs needed to compute a four-point FFT is 8.

FFT Complexity Analysis: $N = 8$

Eight-point FFT computed using radix-2 decimation in time follows the same accounting:

$$X[k] = X_{even}[k] + e^{-j2\pi \cdot k/8} X_{odd}[k] \qquad \text{for } k = 0, 1, \ldots, 7$$

Computation of
$\{X_{even}[0], X_{even}[1], X_{even}[2], X_{even}[3]\} = \mathsf{FFT}_4\{x[0], x[2], x[4], x[6]\}$
requires how many MACs?

FFT Complexity Analysis: $N = 8$

Eight-point FFT computed using radix-2 decimation in time follows the same accounting:

$$X[k] = X_{even}[k] + e^{-j2\pi \cdot k/8} X_{odd}[k] \qquad \text{for } k = 0, 1, \dots, 7$$

Computation of
$\{X_{even}[0], X_{even}[1], X_{even}[2], X_{even}[3]\} = \mathsf{FFT}_4\{x[0], x[2], x[4], x[6]\}$
requires how many MACs? 8

Computation of
$\{X_{odd}[0], X_{odd}[1], X_{odd}[2], X_{odd}[3]\} = \mathsf{FFT}_4\{x[1], x[3], x[5], x[7]\}$ requires how many MACs?

FFT Complexity Analysis: $N = 8$

Eight-point FFT computed using radix-2 decimation in time follows the same accounting:

$$X[k] = X_{even}[k] + e^{-j2\pi \cdot k/8} X_{odd}[k] \qquad \text{for } k = 0, 1, \ldots, 7$$

Computation of $\{X_{even}[0], X_{even}[1], X_{even}[2], X_{even}[3]\} = \mathsf{FFT}_4\{x[0], x[2], x[4], x[6]\}$ requires how many MACs? 8

Computation of $\{X_{odd}[0], X_{odd}[1], X_{odd}[2], X_{odd}[3]\} = \mathsf{FFT}_4\{x[1], x[3], x[5], x[7]\}$ requires how many MACs? 8

How many more MACs are required to assemble the results into an eight-point FFT?

FFT Complexity Analysis: $N = 8$

Eight-point FFT computed using radix-2 decimation in time follows the same accounting:

$$X[k] \;=\; X_{even}[k] + e^{-j2\pi \cdot k/8} X_{odd}[k] \qquad \text{for } k = 0, 1, \dots, 7$$

Computation of $\{X_{even}[0], X_{even}[1], X_{even}[2], X_{even}[3]\} = \text{FFT}_4\{x[0], x[2], x[4], x[6]\}$ requires how many MACs? 8

Computation of $\{X_{odd}[0], X_{odd}[1], X_{odd}[2], X_{odd}[3]\} = \text{FFT}_4\{x[1], x[3], x[5], x[7]\}$ requires how many MACs? 8

How many more MACs are required to assemble the results into an eight-point FFT? 8

Hence, the total MACs needed to compute an eight-point FFT is 24.

FFT Complexity Analysis: General N

OPERATION	MACs
Computation of N one-point FFTs	0
Assembling $\frac{N}{2}$ two-point FFTs from N one-point FFTs	N
Assembling $\frac{N}{4}$ four-point FFTs from $\frac{N}{2}$ two-point FFTs	N
Assembling $\frac{N}{8}$ eight-point FFTs from $\frac{N}{4}$ four-point FFTs	N
⋮	⋮
Assembling one N-point FFT from two $\frac{N}{2}$-point FFTs	N

Hence, the total MACs needed to compute an N-point FFT is ______.