ECE503: Digital Signal Processing Lecture 4

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Lecture 4 Topics

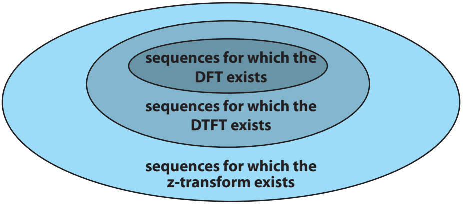
1. Motivation for the z-transform.
2. Definition
3. Region of Convergence
4. Relationship with the DTFT
5. Poles and zeros
6. Inverse z-transform
7. Convolution theorem
8. Transfer functions
9. Stability criterion

Don't We Already Have Enough Transforms?

We already have the DTFT, DFT, DCT, ... Why do we need another transform?

1. DFT/DCT only applicable for finite-length signals.
2. DTFT doesn't uniformly converge for lots of interesting signals, e.g.

$$\text{DTFT}(\mu[n]) = \sum_{n=0}^{\infty} e^{-j\omega n} = ? \quad \text{(not absolutely summable)}$$



Other useful things about the z-transform:

- We can solve for the output of certain types of systems algebraically.
- We can easily determine the stability of a system.

The z-Transform and its Region of Convergence

Definition (bilateral z-transform):

$$\mathcal{Z}(\{x[n]\}) = X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

where $z \in \mathbb{C}$. The set of values $z \in \mathcal{S} \subset \mathbb{C}$ for which this sum converges is called the "region of convergence" (ROC).

Since z is a complex number, it has a magnitude and phase, i.e. $z = re^{j\omega}$. Hence

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] \underbrace{r^{-n}e^{-j\omega n}}_{z^{-n}} = \sum_{n=-\infty}^{\infty} \underbrace{x[n]r^{-n}}_{g[n]} e^{-j\omega n} = \sum_{n=-\infty}^{\infty} g[n]e^{-j\omega n}$$

the z-transform can be thought of as the DTFT of the modified sequence $g[n] = x[n]r^{-n}$. Even in cases when the DTFT of $x[n]$ doesn't exist, the DTFT of $g[n]$ may exist for some values of $z \in \mathbb{C}$ if $\mathcal{S} \neq \emptyset$.

Region of Convergence

Formally, we define the region of convergence $\mathcal{S} \subset \mathbb{C}$ of the sequence $\{x[n]\}$ as

$$\mathcal{S} = \left\{ z \in \mathbb{C} \,:\, \sum_{n=-\infty}^{\infty} |x[n]z^{-n}| < \infty \right\}$$

Remarks:

- Suppose you know the sum above converges for a particular $z_1 = r_1 e^{j\omega_1}$. Then it converges for all z with $|z| = |z_1| = r_1$.
- The ROC is important because different sequences can have the same z-transform, i.e. the z-transform is not unique without its ROC.
- When we specify the Z-transform of a sequence, we also must specify its ROC (except for certain special cases):

$$x[n] \stackrel{\mathcal{Z}}{\longleftrightarrow} X(z) \qquad \text{ROC} : \mathcal{S}$$

Region of Convergence

Example 1: $x[n] = \mu[n]$. The ROC is all $z \in \mathbb{C}$ such that $\sum_{n=0}^{\infty} |z|^{-n} < \infty$. We know this sum is finite only if $|z| > 1$. Hence the ROC of $x[n] = \mu[n]$ is $\mathcal{S} = \{z \in \mathbb{C} : |z| > 1\}$. For $z \in \mathcal{S}$, we have

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = \sum_{n=0}^{\infty} z^{-n} = \sum_{n=0}^{\infty} (z^{-1})^n = \frac{1}{1-z^{-1}}.$$

Example 2: $x[n] = -\mu[-n-1]$. The ROC is all $z \in \mathbb{C}$ such that $\sum_{n=-\infty}^{-1} |z|^{-n} = \sum_{n=1}^{\infty} |z|^{n} < \infty$. We know this sum is finite only if $|z| < 1$. Hence the ROC of $x[n] = -\mu[-n-1]$ is $\mathcal{S} = \{z \in \mathbb{C} : |z| < 1\}$. For $z \in \mathcal{S}$, we have

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = -\sum_{n=-\infty}^{-1} z^{-n} = -\sum_{n=1}^{\infty} z^{n} = -\frac{z}{1-z} = \frac{1}{1-z^{-1}}.$$

Same $X(z)$ but different ROC.

Region of Convergence

Example 3: $x[n] = \alpha^n$ for $\alpha \in \mathbb{C}$. We can write

$$\begin{aligned} X(z) &= \sum_{n=-\infty}^{\infty} x[n] z^{-n} \\ &= \sum_{n=-\infty}^{-1} \alpha^n z^{-n} + \sum_{n=0}^{\infty} \alpha^n z^{-n} \\ &= \sum_{n=1}^{\infty} \alpha^{-n} z^n + \sum_{n=0}^{\infty} \alpha^n z^{-n} \\ &= \sum_{n=1}^{\infty} (\alpha^{-1} z)^n + \sum_{n=0}^{\infty} (\alpha z^{-1})^n \end{aligned}$$

The first sum is finite for what values of $z \in \mathbb{C}$?

The second sum is finite for what values of $z \in \mathbb{C}$?

What can we say about the ROC?

The z-Transform and the DTFT

Recall

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} \qquad \text{and} \qquad X(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

It is clear that the DTFT is a special case of the z-transform with $z = e^{j\omega}$.

The DTFT exists if and only if the ROC of $X(z)$ includes the ring $|z| = 1$. It is incorrect to just substitute $X(\omega) = X(z)|_{z=e^{j\omega}}$ if the ROC of $X(z)$ does not include the ring $|z| = 1$.

Example: We saw earlier that, given $x[n] = \mu[n]$, we can compute $X(z) = \frac{1}{1-z^{-1}}$. Does $X(\omega) = \frac{1}{1-e^{-j\omega}}$?

Rational z-Transforms: Poles and Zeros

Most sequences of interest have rational z-transforms (see Table 6.1 on p. 281) with the following form

$$X(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_M z^{-M}}{1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_N z^{-N}}$$
$$= b_0 z^{(N-M)} \frac{\prod_{m=1}^{M}(z - \xi_m)}{\prod_{n=1}^{N}(z - \lambda_n)} = b_0 z^{(N-M)} \frac{P(z)}{Q(z)}$$

where $P(z)$ and $Q(z)$ are polynomials in z.

Definition

The **zeros** of $X(z)$ are the set of values of $z \in \mathbb{C}$ such that $X(z) = 0$.

Definition

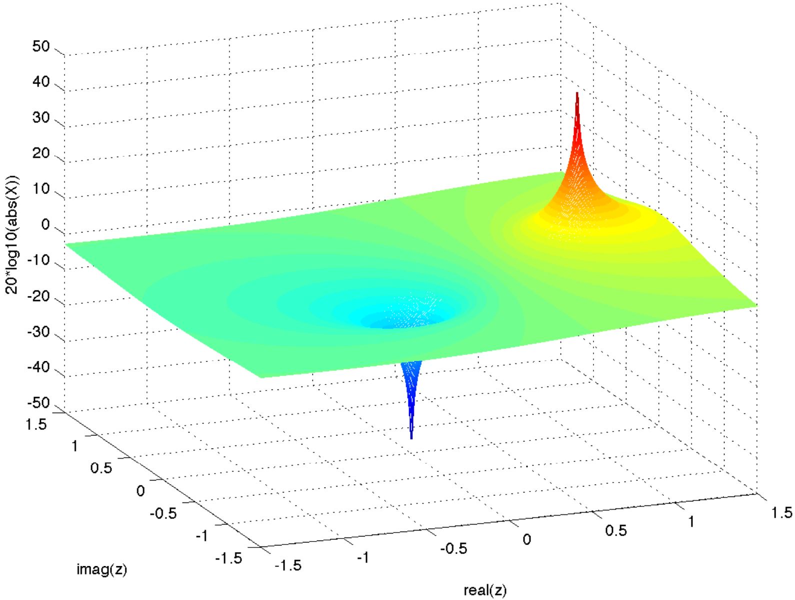
The **poles** of $X(z)$ are the set of values of $z \in \mathbb{C}$ such that $X(z) = \pm\infty$.

Rational z-Transforms: Poles and Zeros

Example: $X(z) = \frac{1}{1-z^{-1}}$ with ROC $|z| > 1$.

What are the poles?

What are the zeros?


50
40
30
20
10
0
-10
-20
-30
-40
-50
20*log10(abs(X))
1.5
1
0.5
0
-0.5
-1
-1.5
imag(z)
-1.5
-1
-0.5
0
0.5
1
1.5
real(z)

Rational z-Transforms: Poles and Zeros

For sequences with a rational z-transform, we have:

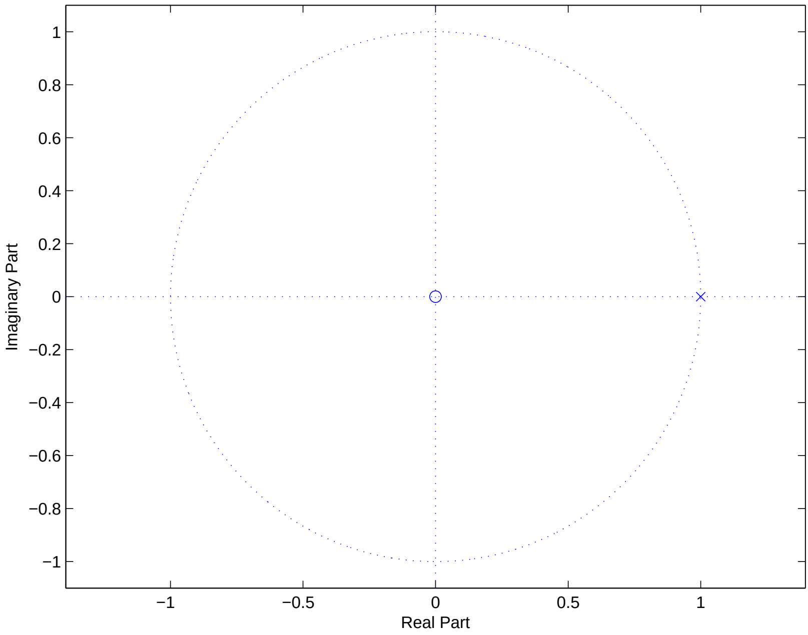
$$X(z) = b_0 z^{(N-M)} \frac{\prod_{m=1}^{M}(z-\xi_m)}{\prod_{n=1}^{N}(z-\lambda_n)} = b_0 z^{(N-M)} \frac{P(z)}{Q(z)}$$

Remarks:

- If $P(z)$ and $Q(z)$ are coprime, then the finite zeros of $X(z)$ are the roots of $P(z)$ and the finite poles of $X(z)$ are the roots of $Q(z)$.
- If $N > M$ there will be $N - M$ additional zeros at $z = 0$.
- If $N < M$, there will be $M - N$ additional poles at $z = 0$.

Matlab can easily convert from the coefficients of a rational z-transform to the pole/zeros factorization, e.g. `[z,p,k] = tf2zpk(num,den)`.

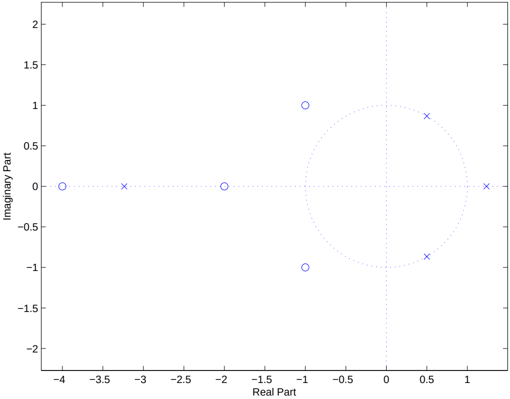
Other potentially useful Matlab functions: `roots`, `poly`, `zplane`.


1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1
Imaginary Part
−1
−0.5
0
0.5
1
Real Part

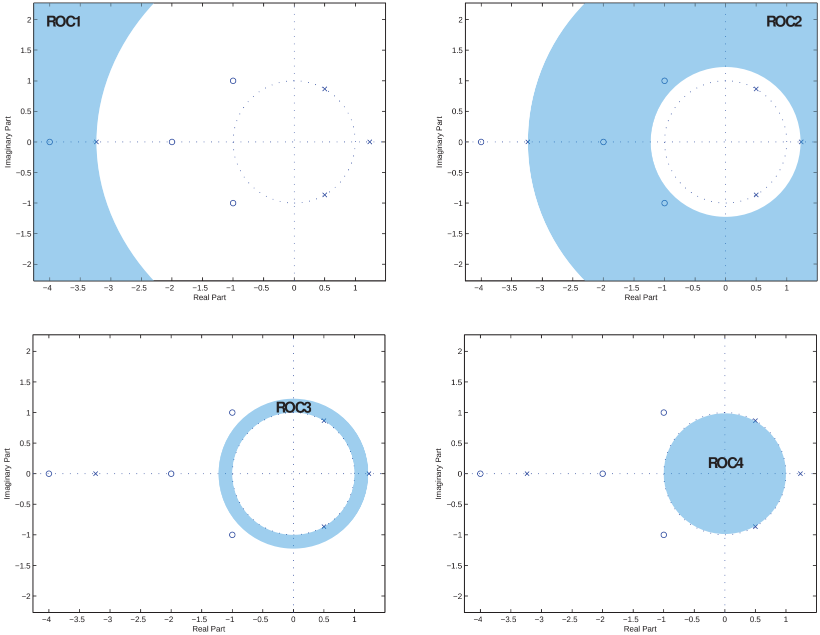
ROC of Rational z-Transforms

It should be clear that the ROC of a rational z-transform $H(z)$ can't contain a pole. For example, suppose $X(z) = \frac{2z^4+16z^3+44z^2+56z+32}{3z^4+3z^3-15z^2+18z-12}$. Then

```
b = [2,16,44,56,32];
a = [3,3,-15,18,-12];
zplane(b,a);
```



Note the poles are $\lambda_1 = -3.2361$, $\lambda_2 = 1.2361$, $\lambda_3 = 0.5000 - j0.8660$, and $\lambda_4 = 0.5000 + j0.8660$. What are the possible ROCs for this $X(z)$?


ROC1
ROC2
ROC3
ROC4
Imaginary Part
Real Part

ROC Properties for Rational z-Transforms (1 of 2)

1. The ROC is a ring or a disk in the z-plane centered at the origin.
2. The ROC cannot contain any poles.
3. If $\{x[n]\}$ is a **finite-length** sequence, then the ROC is the entire z-plane except possibly $z = 0$ or $|z| = \infty$.
4. If $\{x[n]\}$ is an **infinite-length right-sided** sequence, then the ROC extends outward from the largest magnitude finite pole of $X(z)$ to (and possibly including) $|z| = \infty$.
5. If $\{x[n]\}$ is an **infinite-length left-sided** sequence, then the ROC extends inward from the smallest magnitude finite pole of $X(z)$ to (and possibly including) $z = 0$.
6. If $\{x[n]\}$ is an **infinite-length two-sided** sequence, then the ROC will be a ring on the z-plane, bounded on the interior and exterior by a pole, and not containing any poles.

ROC Properties for Rational z-Transforms (2 of 2)

7. The ROC must be a connected region.
8. The DTFT of the sequence $\{x[n]\}$ converges absolutely if and only if the ROC of $X(z)$ contains the unit circle.
9. If $\{x[n]\} \stackrel{\mathcal{Z}}{\longleftrightarrow} X(z)$ with ROC:$\mathcal{S}_X$ and $\{y[n]\} \stackrel{\mathcal{Z}}{\longleftrightarrow} Y(z)$ with ROC:$\mathcal{S}_Y$, then the sequence $\{u[n]\} = \{ax[n] + by[n]\}$ will have a z-transform $\{u[n]\} \stackrel{\mathcal{Z}}{\longleftrightarrow} aX(z) + bY(z)$ with ROC that includes $\mathcal{S}_X \bigcap \mathcal{S}_Y$.

Note, in property 9, the ROC of $U(z) = aX(z) + bY(z)$ can be bigger than $\mathcal{S}_X \bigcap \mathcal{S}_Y$. For example:

$$x[n] = \mu[n] \stackrel{\mathcal{Z}}{\longleftrightarrow} X(z) = \frac{1}{1 - z^{-1}} \quad \text{ROC} : |z| > 1$$

$$y[n] = \mu[n-1] \stackrel{\mathcal{Z}}{\longleftrightarrow} Y(z) = \frac{z^{-1}}{1 - z^{-1}} \quad \text{ROC} : |z| > 1$$

$$u[n] = x[n] - y[n] = \delta[n] \stackrel{\mathcal{Z}}{\longleftrightarrow} U(z) = X(z) - Y(z) = 1 \quad \text{ROC} : \text{all } z$$

Inverse z-Transform

The inverse z-transform is based on a special case of the Cauchy integral theorem

$$\frac{1}{2\pi j}\oint_C z^{-\ell}\,dz = \begin{cases} 1 & \ell = 1 \\ 0 & \ell \neq 1 \end{cases}$$

where C is a counterclockwise contour that encircles the origin. If we multiply $X(z)$ by z^{n-1} and compute

$$\begin{aligned} \frac{1}{2\pi j}\oint_C X(z)z^{n-1}\,dz &= \frac{1}{2\pi j}\oint_C \sum_{m=-\infty}^{\infty} x[m]z^{-m+n-1}\,dz \\ &= \sum_{m=-\infty}^{\infty} x[m]\underbrace{\frac{1}{2\pi j}\oint_C z^{-(m-n+1)}\,dz}_{=1 \text{ only when } m-n+1=1} \\ &= \sum_{m=-\infty}^{\infty} x[m]\delta(m-n) \\ &= x[n] \end{aligned}$$

Hence, the inverse z-transform of $X(z)$ is defined as $x[n] = \frac{1}{2\pi j}\oint_C X(z)z^{n-1}\,dz$ where C is a counterclockwise closed contour in the ROC of $X(z)$ encircling the origin.

Inverse z-Transform via Cauchy's Residue Theorem

Denote the unique poles of $X(z)$ as $\lambda_1, \dots, \lambda_R$ and their algebraic multiplicities as $m_1, \dots, m_R$. As long as R is finite (which is the case if $X(z)$ is rational) we can evaluate the inverse z-transform via Cauchy's residue theorem which states

$$x[n] = \frac{1}{2\pi j} \oint_C X(z) z^{n-1}\, dz = \sum_{\lambda_k \text{ inside } C} \mathrm{Res}(X(z)z^{n-1}, \lambda_k, m_k)$$

where $\mathrm{Res}(F(z), \lambda_k, m_k)$ is the "residue" of $F(z) = X(z)z^{n-1}$ at the pole λ_k with algebraic multiplicity m_k, defined as

$$\mathrm{Res}(F(z), \lambda_k, m_k) = \frac{1}{(m_k - 1)!} \left[\frac{d^{m_k - 1}}{dz^{m_k - 1}} \left\{ (z - \lambda_k)^{m_k} F(z) \right\} \right]_{z = \lambda_k}$$

In other words, Cauchy's residue theorem allows us to compute the contour integral by computing derivatives.

Inverse z-Transform via Cauchy's Residue Theorem

Simple example: Suppose $X(z) = \frac{1}{1-az^{-1}}$ with ROC $|z| > |a|$.

What are the poles of $X(z)$? $\lambda_1 = a$ and $m_1 = 1$.

Now what are the poles of $X(z)z^{n-1}$?

- For $n = 0$, $X(z)z^{n-1} = \frac{z^{-1}}{1-az^{-1}} = \frac{1}{z-a}$. One pole at $z = a$.
- For $n = 1, 2, \dots,$ $X(z)z^{n-1} = \frac{z^{n-1}}{1-az^{-1}} = \frac{z^n}{z-a}$. Still one pole at $z = a$.
- For $n = -1, -2, \dots,$ $X(z)z^{n-1} = \frac{z^{n-1}}{1-az^{-1}} = \frac{1}{z^{-n}(z-a)}$. One pole at $z = a$ and now also $-n$ poles at $z = 0$.

For $n = 0, 1, \dots,$ we can write

$$x[n] = \frac{1}{2\pi j} \oint_C X(z)z^{n-1}\,dz = \frac{1}{2\pi j} \oint_C \frac{z^{n-1}}{1 - az^{-1}}\,dz$$
$$= \frac{1}{0!} \left[\frac{d^0}{dz^0} \left\{ (z-a) \frac{z^{n-1}}{1 - az^{-1}} \right\} \right]_{z=a} = [z^n]_{z=a} = a^n$$

Continued...

Inverse z-Transform via Cauchy's Residue Theorem

For negative values of n, we have a second pole $\lambda_2 = 0$ with algebraic multiplicity $m_2 = -n$. We can write

$$\begin{aligned} x[n] &= \frac{1}{2\pi j}\oint_C X(z)z^{n-1}\,dz \\ &= \frac{1}{2\pi j}\oint_C \frac{z^{n-1}}{1-az^{-1}}\,dz \\ &= \left[\frac{d^0}{dz^0}\left\{(z-a)\frac{z^{n-1}}{1-az^{-1}}\right\}\right]_{z=a} + \frac{1}{(-n-1)!}\left[\frac{d^{-n-1}}{dz^{-n-1}}\left\{(z-0)^{-n}\frac{z^{n-1}}{1-az^{-1}}\right\}\right]_{z=0} \\ &= a^n + \frac{1}{(-n-1)!}\left[\frac{d^{-n-1}}{dz^{-n-1}}\left\{\frac{1}{z-a}\right\}\right]_{z=0} \end{aligned}$$

- For $n = -1$, the second residue is simply $\frac{1}{0!}(1/(0-a)) = -a^{-1}$.
- For $n = -2$, the second residue is $\frac{1}{1!}\left[\frac{d}{dz}\left\{\frac{1}{z-a}\right\}\right]_{z=0} = -(z-a)^{-2}|_{z=0} = -a^{-2}$.
- For $n = -3$, the second residue is $\frac{1}{2!}\left[\frac{d^2}{dz^2}\left\{\frac{1}{z-a}\right\}\right]_{z=0} = (z-a)^{-3}|_{z=0} = -a^{-3}$.
- For general $n < 0$, the second residue can be computed as $-a^n$.

Hence $x[n] = 0$ for all $n < 0$.

Other Methods for Computing Inverse z-Transforms

Cauchy's residue theorem works, but it can be tedious and there are lots of ways to make mistakes. The Matlab function `residuez` (discrete-time residue calculator) can be useful to check your results.

Here are some other options for computing inverse z-transforms:

1. Inspection (table lookup).
2. Partial fraction expansion (only for rational z-transforms).
3. Power series expansion (can be used for non-rational z-transforms).

I'll do examples of each of these.

The Matlab function `residuez` is also useful in partial fraction expansions of rational $X(z)$.

Convolution Theorem

You should familiarize yourself with the theorems in Section 6.5 of your textbook (in particular, how the ROC is affected). A particularly important theorem for z-transforms is the convolution theorem:

Theorem

If $\{x[n]\} \stackrel{\mathcal{Z}}{\longleftrightarrow} X(z)$ *with ROC:*$\mathcal{S}_X$ *and* $\{y[n]\} \stackrel{\mathcal{Z}}{\longleftrightarrow} Y(z)$ *with ROC:*$\mathcal{S}_Y$*, then the sequence* $\{u[n]\} = \{x[n]\} \circledast \{y[n]\}$ *will have a* z *transform* $\{u[n]\} \stackrel{\mathcal{Z}}{\longleftrightarrow} X(z)Y(z)$ *with ROC including* $\mathcal{S}_X \bigcap \mathcal{S}_Y$.

Note, just like the linearity property, the ROC of $U(z) = X(z)Y(z)$ can be bigger than $\mathcal{S}_X \bigcap \mathcal{S}_Y$. See example 6.28 in your textbook.

For an LTI system $\mathcal{H}$ with impulse response $\{h[n]\}$, we have $y[n] = h[n] \circledast x[n]$, hence

$$Y(z) = H(z)X(z) \text{ with ROC: } \mathcal{S}_Y$$

where $H(z)$ is the z-transform of the impulse response $\{h[n]\}$ and is commonly called the transfer function of the LTI system $\mathcal{H}$.

Transfer Function from a Finite-Dimensional Difference Eq.

Most LTI systems of practical interest can be described by finite-dimensional constant-coefficient difference equations, e.g.

$$y[n] = \sum_{k=0}^{M-1} b_k x[n-k] - \sum_{k=1}^{N-1} a_k y[n-k]$$

Even though this system is causal, we don't require causality in the following analysis. We can take the z-transform of both sides using the time-shifting property of the z-transform to write

$$Y(z) = \sum_{k=0}^{M-1} b_k z^{-k} X(z) - \sum_{k=1}^{N-1} a_k z^{-k} Y(z)$$

and group terms to write

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M-1} b_k z^{-k}}{\sum_{k=1}^{N-1} a_k z^{-k}}.$$

From this result (and knowing the ROC), you can calculate the inverse z-transform to get the impulse response $\{h[n]\}$. This fully describes the relaxed behavior (zero state response) of the LTI system.

Transfer Function ROC

Recall the ROC properties discussed earlier. If we know certain things about the system with transfer function $H(z)$, we can apply our earlier results to specify the ROC of the transfer function as follows:

- If the transfer function only has poles at zero (corresponding to a finite-length impulse response), then its ROC is all $|z| > 0$.
- If the transfer function corresponds to a **causal** system and has poles not at zero (corresponding to an infinite-length impulse response), then the ROC extends outward from the largest magnitude finite pole of $X(z)$ to (and possibly including) $|z| = \infty$.
- If the transfer function corresponds to a **anti-causal** system and has poles not at zero (corresponding to an infinite-length impulse response), then the ROC extends inward from the smallest magnitude finite pole of $X(z)$ to (and possibly including) $z = 0$.

Transfer Function Description: Capabilities and Limitations

- \+ Can describe memoryless or dynamic systems.
- \+ Can describe causal and non-causal systems (ROC).
- – Not useful for non-linear systems.
- – Not useful for time-varying systems.
- – No explicit access to internal behavior of system.
- – Can't describe systems with non-zero initial conditions. Implicitly assumes that system is relaxed.
- \+ Abundance of analysis techniques. Systems are usually analyzed with **basic algebra**, not calculus.

Determining Stability from the Transfer Function

Definition

A discrete-time system is BIBO stable if, for every input satisfying

$$|x[k]| \leq M_x$$

for all $k \in \mathbb{Z}$ and some $0 \leq M_x < \infty$, the output satisfies

$$|y[k]| \leq M_y$$

for all $k \in \mathbb{Z}$ and some $0 \leq M_y < \infty$.

We know that an LTI system $\mathcal{H}$ is BIBO stable if and only if its impulse response $\{h[n]\}$ is absolutely summable (Sec. 4.4.3), i.e.

$$\sum_{n=-\infty}^{\infty} |h[n]| < \infty.$$

This can be tricky to check. Is there an easier test using the transfer function?

Determining Stability from the Transfer Function

Observe that

$$\text{BIBO stable} \quad \Longleftrightarrow \quad \sum_{n=-\infty}^{\infty} |h[n]| < \infty \quad \Longleftrightarrow \quad \sum_{n=-\infty}^{\infty} |h[n]z^{-n}| < \infty$$

for $|z| = 1$. Hence an LTI system is BIBO stable if and only if the ROC of $H(z)$ includes the unit circle. This condition also ensures the DTFT exists.

The rule you probably learned as an undergraduate student is that "an LTI system $\mathcal{H}$ is BIBO stable if and only if all of the poles of $H(z)$ are inside the unit circle". Does this agree with the condition above?

Example: Suppose

$$H(z) = \frac{1}{1 - 2z^{-1}} \quad \text{ROC}: |z| < 2$$

Is this system BIBO stable? What is $\{h[n]\}$?

Conclusions

1. This concludes Chapter 6. You are responsible for all of the material in this chapter, even if it wasn't covered in lecture.
2. Please read Chapter 7 before the next lecture and have some questions prepared.
3. The next lecture is on Monday 13-Feb-2012 at 6pm.