ECE503: Digital Signal Processing
Lecture 5

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WPI

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Lecture 5 Topics

1. Magnitude and phase characterization of transfer functions
2. Linear phase FIR filters
3. Simple FIR and IIR filtering
4. Complementary transfer functions
5. Inverse systems and equalization
6. System identification

Magnitude Response Characterization of Transfer Function

Definition

A causal stable system $\mathcal{H}$ with real-coefficient transfer function $H(z)$ is called **bounded real** (BR) if its DTFT satisfies $|H(\omega)| \leq 1$ for all ω.

Note that causal stable transfer functions have all poles inside the unit circle, hence they must have a bounded magnitude response $|H(\omega)| \leq K$ for all ω. The TF can be scaled by $1/K$ to make the system BR.

Definition

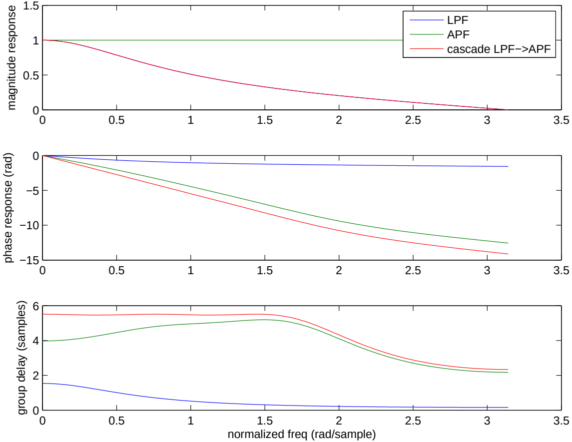
A stable system $\mathcal{H}$ with IIR transfer function $H(z)$ is called **allpass** if its DTFT satisfies $|H(\omega)| = 1$ for all ω.

Definition

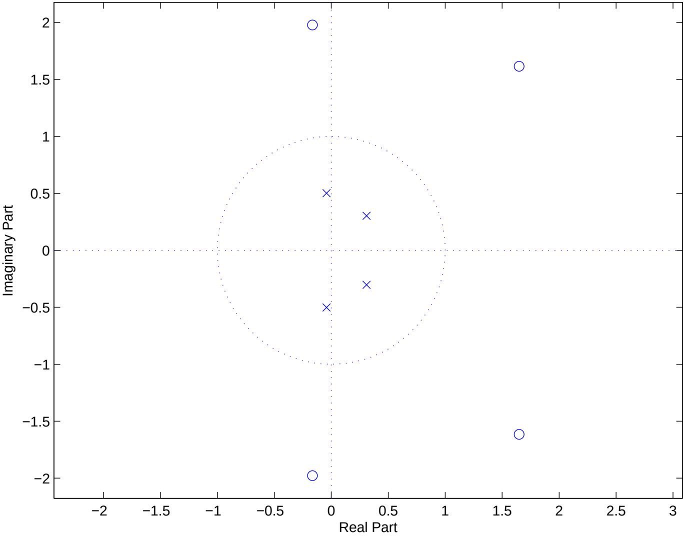
A causal stable system $\mathcal{H}$ with real-coefficient allpass transfer function is called **lossless bounded real** (LBR).

Allpass Delay Equalizer Example

It is not possible to design a stable causal IIR filter with linear phase. So what we can do instead is cascade an allpass filter with an IIR filter to get approximately linear phase over a desired range of frequencies. Example:



Allpass Filter Poles and Zeros Mirrored Across Unit Circle



Phase Response Characterization of Transfer Function

Definition

A stable system $\mathcal{H}$ is called **zero-phase** if it has a DTFT satisfying $\angle H(\omega) = 0$ for all ω where $|H(\omega)| > 0$.

Remark: A zero-phase dynamic system can not be implemented causally. Matlab `filtfilt` is an example of a non-causal zero-phase filtering technique.

Definition

A **linear phase system** $\mathcal{H}$ is a system with phase response $\theta(\omega) = \angle H(\omega) = c\omega$ for all ω and any constant c. An **affine phase system** (also called a generalized linear phase system) $\mathcal{H}$ is a system with phase response $\theta(\omega) = \angle H(\omega) = c\omega + \beta$ for all ω and any constants c and β.

It is always possible to design an FIR filter with linear-phase response. It is not possible to design a stable causal IIR filter with linear phase response.

Phase Response Characterization of Transfer Function

Definition

A causal stable system $\mathcal{H}$ with transfer function $H(z)$ with all zeros inside the unit circle is called **minimum phase**.

Definition

A causal stable system $\mathcal{H}$ with transfer function $H(z)$ with all zeros outside the unit circle is called **maximum phase**.

Definition

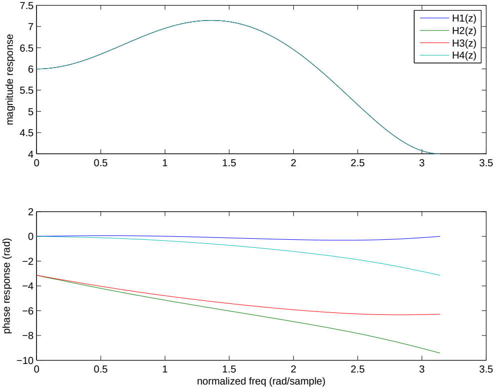
A causal stable system $\mathcal{H}$ with transfer function $H(z)$ with at least one zero inside the unit circle and at least one zero outside the unit circle is called **mixed phase**.

Minimum phase systems are important because they have a stable inverse $G(z) = 1/H(z)$. You can convert between min/max/mixed-phase systems by cascading allpass filters.

Phase Response Characterization Example

$$H_1(z) = 6 + z^{-1} - z^{-2} \qquad H_2(z) = 1 - z^{-1} - 6z^{-2}$$
$$H_3(z) = 2 - 5z^{-1} - 3z^{-2} \qquad H_4(z) = 3 + 5z^{-1} - 2z^{-2}$$

As shown below, all four of these causal stable systems have the same magnitude response. Which are minimum, maximum, and mixed phase?



Four Types of Linear-Phase FIR Filters (1 of 2)

Type	Impulse response symmetry	Impulse response length
I	symmetric	N odd (order is even)
II	symmetric	N even (order is odd)
III	antisymmetric	N odd (order is even)
IV	antisymmetric	N even (order is odd)

Symmetric length-N impulse response: $h[n] = h[N-1-n]$.
Antisymmetric length-N impulse response: $h[n] = -h[N-1-n]$.

All of these types will have frequency response of the form

$$H(\omega) = e^{-j(N-1)\omega/2} e^{j\beta} \breve{H}(\omega)$$

where $\breve{H}(\omega) : \mathbb{R} \mapsto \mathbb{R}$ is called the **amplitude response**.

Note $N-1$ is the **order** of the system (this can be confusing). Also note these filters all have the same constant group delay: $\tau_g(\omega) = \frac{(N-1)}{2}$.

Four Types of Linear-Phase FIR Filters (2 of 2)

Type	LPF	HPF	BPF	BSF	Comment
I	Y	Y	Y	Y	Most versatile.
II	Y	N	Y	N	Zero at $z = -1$.
III	N	N	Y	N	Zeros at $z = \pm 1$.
IV	N	Y	Y	N	Zero at $z = 1$.

Simple Filtering I: Lowpass FIR

Length $N = 2$ moving average filter just takes the average of the last two inputs:

$$H(z) = \frac{1}{2}(1 + z^{-1}) = \frac{z+1}{2z}$$

Since this is an FIR filter, it has ROC everywhere except $z = 0$.

$$H(\omega) = \frac{1}{2}(1 + e^{-j\omega}) = e^{-j\omega/2}\cos(\omega/2)$$

Linear phase?

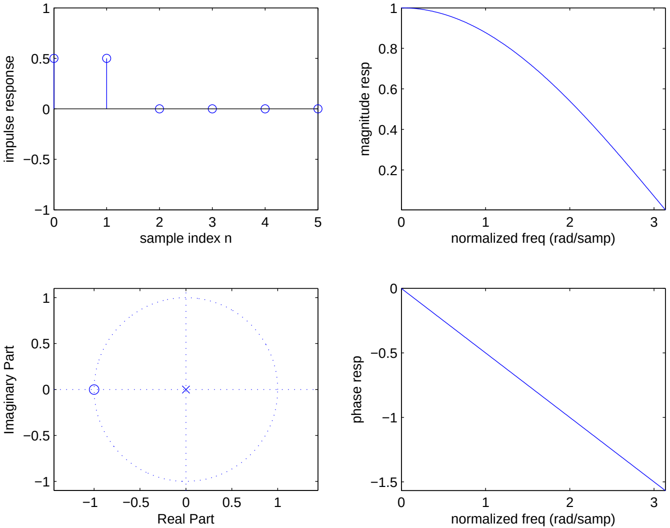
Type?

Cutoff frequency is defined as the value of ω such that $|H(\omega)|^2 = \frac{1}{2}$. This is easy to compute:

$$\cos^2(\omega_c/2) = 1/2 \qquad \Longleftrightarrow \qquad \omega_c = \pi/2.$$

Can cascade to decrease width of passband.

Simple Filtering I: Lowpass FIR



Simple Filtering II: Highpass FIR

Length $N = 2$ "moving difference" filter just takes the difference of the last two inputs:

$$H(z) = \frac{1}{2}(1 - z^{-1}) = \frac{z-1}{2z}$$

Since this is an FIR filter, it has ROC everywhere except $z = 0$.

$$H(\omega) = \frac{1}{2}(1 - e^{-j\omega}) = je^{-j\omega/2}\sin(\omega/2)$$

Linear phase?

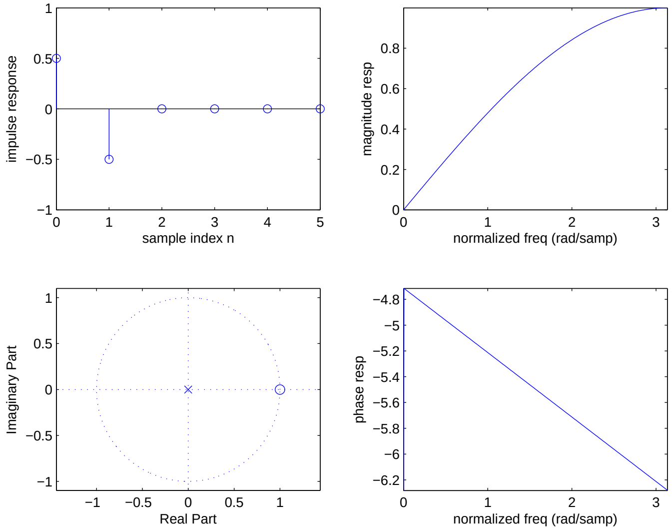
Type?

Cutoff frequency is easy to compute:

$$\sin^2(\omega_c/2) = 1/2 \qquad \Longleftrightarrow \qquad \omega_c = \pi/2.$$

Can cascade to decrease width of passband.

Simple Filtering II: Highpass FIR



Simple Filtering III: Notch FIR

You can't get a notch with a length-2 FIR filter. We need at least length $N = 3$. This transfer function will work:

$$H(z) = 1 - 2\cos(\omega_0)z^{-1} + z^{-2} = \frac{z^2 - 2\cos(\omega_0)z + 1}{z^2}$$

(note book eq (7.74) is wrong). Not difficult to confirm $z = e^{\pm j\omega_0}$ are the zeros. Since this is an FIR filter, it has ROC everywhere except $z = 0$.

$$H(\omega) = 1 - 2\cos(\omega_0)e^{-j\omega} + e^{-j2\omega} = e^{-j\omega}(2\cos(\omega) - 2\cos(\omega_0))$$

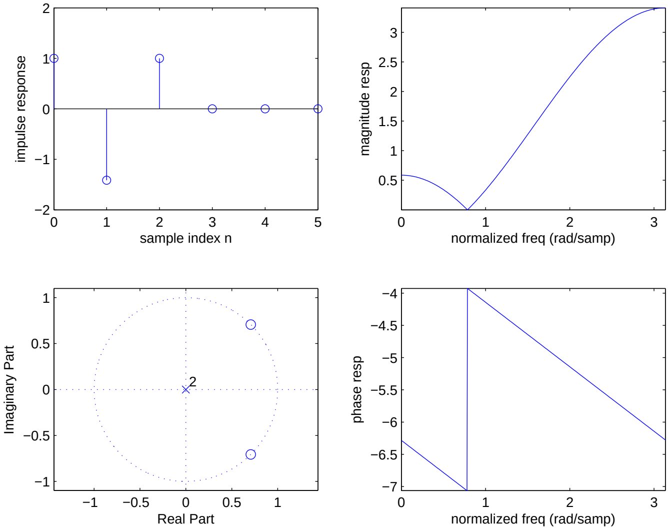
Easy to see $H(\omega) = 0$ when $\omega = \pm\omega_0$.

Linear phase?

Type?

Cascading may not be a good idea for this filter.

Simple Filtering III: Notch FIR



Simple Filtering IV: Lowpass IIR

Intuition:

- We want a zero at $z = -1$ to block the high frequencies.
- We want a pole somewhere near $z = 1$ (but inside the unit circle if our filter is to be causal & stable) to provide gain to the low frequencies.

Candidate transfer function with real $0 < \alpha < 1$:

$$H(z) = \frac{K(1 + z^{-1})}{1 - \alpha z^{-1}} = \frac{1-\alpha}{2}\frac{1 + z^{-1}}{1 - \alpha z^{-1}}$$

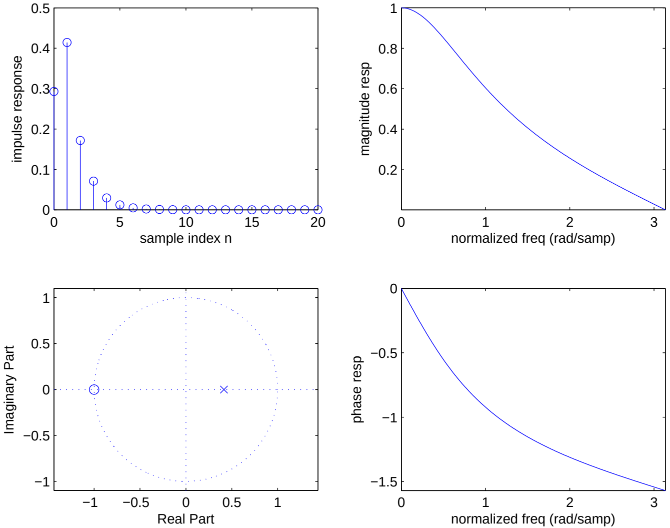
where in the second equality we've selected K so that the maximum magnitude (which occurs at $z = 1$) is one.

We can find the cutoff frequency with a bit of algebra to be $\omega_c = \cos^{-1}\left(\frac{2\alpha}{1+\alpha^2}\right)$, or we can solve for α in terms of the cutoff frequency as

$$\alpha = \frac{1 - \sin\omega_c}{\cos\omega_c}.$$

What happens if we set $\omega_c = \pi/2$?

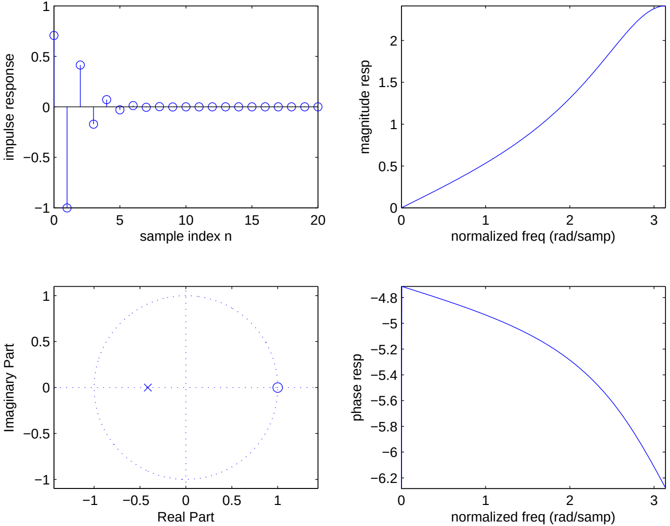
Simple Filtering IV: Lowpass IIR



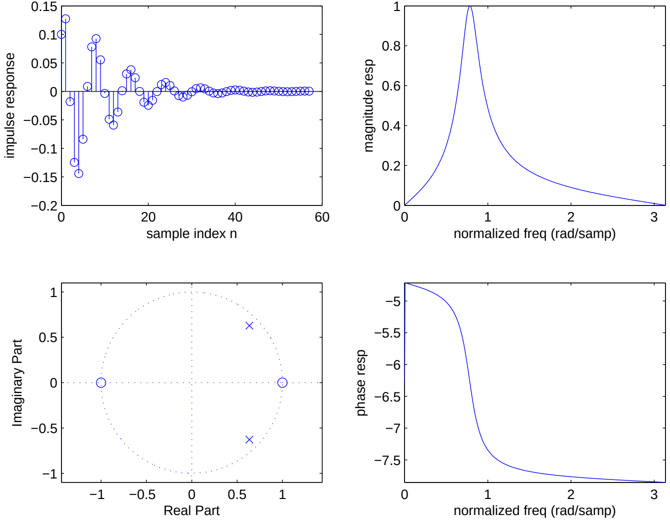
Simple Filtering V: Highpass IIR

Using the intuition you now have about the effect of poles and zeros on the frequency response, how would you construct a simple IIR highpass filter?

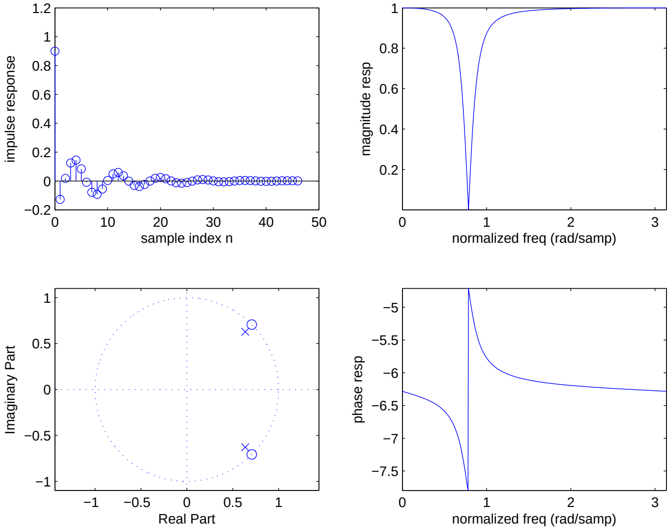
Simple Filtering V: Highpass IIR



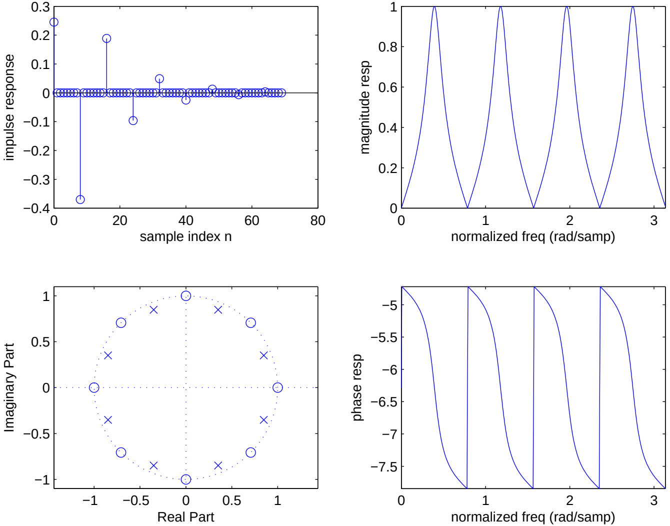
Simple Filtering V: Bandpass IIR



Simple Filtering VI: Bandstop/Notch IIR



Simple Filtering VII: IIR Comb Filter



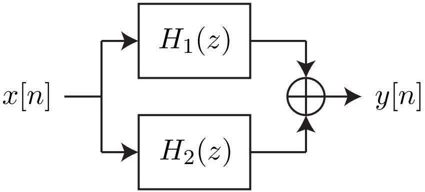
Delay-Complementary Transfer Functions

Definition

A set of L transfer functions $\{H_1(z), \dots, H_L(z)\}$ are called **delay-complementary** if their sum is equal to a scaled integer delay, i.e.

$$\sum_{k=1}^{L} H_k(z) = cz^{-n_0}$$

for $c \neq 0$ and any $n_0 \in \{0, 1, \dots\}$.



If $H_1(z)$ and $H_2(z)$ are delay complementary, then $y[n] = cx[n - n_0]$.

When might something like this be useful?

Allpass-Complementary Transfer Functions

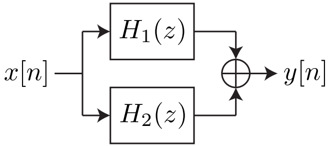
Definition

A set of L stable transfer functions $\{H_1(z), \ldots, H_L(z)\}$ are called **allpass-complementary** if their sum is equal to an allpass transfer function, i.e.

$$\sum_{k=1}^{L} H_k(z) = \mathcal{A}(z)$$

where $|\mathcal{A}(\omega)| = 1$ for all ω.

Delay complementary transfer functions with $c = 1$ are also allpass complementary. Some allpass complementary transfer functions are also delay complementary.



If $H_1(z)$ and $H_2(z)$ are allpass complementary, then $Y(\omega) = X(\omega)e^{jf(\omega)}$ for some phase function $f(\omega)$.

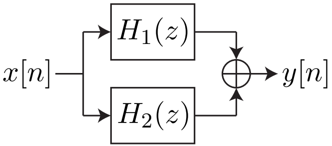
Power-Complementary Transfer Functions

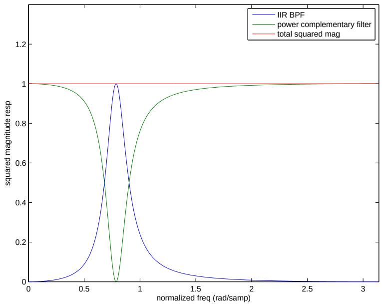
Definition

A set of L stable transfer functions $\{H_1(z), \dots, H_L(z)\}$ are called **power-complementary** if their squared magnitude sum is a constant for all ω, i.e.

$$\sum_{k=1}^{L} |H_k(\omega)|^2 = c > 0 \qquad \text{for all } \omega.$$

See Matlab function `iirpowcomp`.





Magnitude and Doubly-Complementary Transfer Functions

Definition

A set of L stable transfer functions $\{H_1(z), \dots, H_L(z)\}$ are called **magnitude-complementary** if their magnitude sum is a constant for all ω, i.e.

$$\sum_{k=1}^{L} |H_k(\omega)| = c > 0 \qquad \text{for all } \omega.$$

Same idea as power-complementary except we are adding magnitudes here, not squared magnitudes.

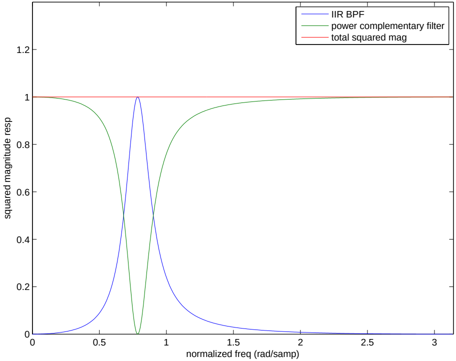
Definition

A set of L stable transfer functions $\{H_1(z), \dots, H_L(z)\}$ satisfying the allpass-complementary property and the power-complementary property are called **doubly-complementary**.

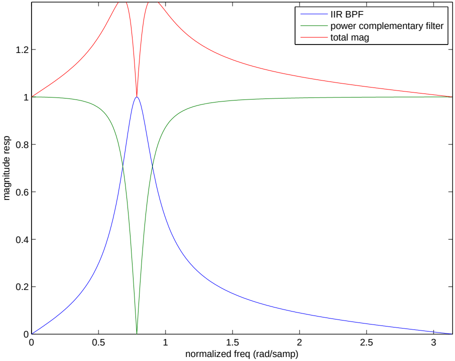
Power-Complementary vs. Magnitude Complementary

```
omega0 = pi/4;                 % bandpass filter center frequency
beta = cos(omega0);
alpha = 0.8;                   % controls bandwidth of BPF
b = (1-alpha)/2*[1 0 -1];      % numerator
a = [1 -beta*(1+alpha) alpha]; % denominator
[bp,ap] = iirpowcomp(b,a);     % compute power complementary filter
```

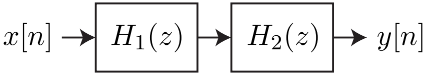
squared-magnitude plot



magnitude plot



Inverse Systems and Equalization



The system $H_2(z)$ is the inverse of $H_1(z)$ if $y[n] = x[n]$. This is equivalent to saying $H_1(z)H_2(z) = 1$ or $h_1[n] \circledast h_2[n] = \delta[n]$.

Remarks:

1. When might something like this be useful?
2. Note that the zeros of $H_1(z)$ become the poles of $H_2(z)$.
3. Note that the inverse system usually does not have a unique impulse response unless you further constrain the inverse system to be causal and/or stable (which identifies the ROC).
4. It is often the case that a causal stable inverse can not be found. One workaround is to find a causal stable **generalized inverse** such that $H_1(z)H_2(z) = z^{-n_0}$ for some integer n_0.

Equalization of Nonminimum Phase Channel

Suppose $H_1(z) = \frac{(z-4)(z+5)}{(z+0.5)(z-0.3)}$ with ROC $|z| > 0.5$.

We form the inverse system $H_2(z) = \frac{(z+0.5)(z-0.3)}{(z-4)(z+5)}$. What are the possible ROCs? Is there a causal stable inverse?

One approach in this case is to factor $H_1(z)$ into a causal stable minimum phase filter and a causal stable allpass filter, i.e.

$$H_1(z) = H_{min}(z)\mathcal{A}(z) = \underbrace{\frac{(4z-1)(5z+1)}{(z+0.5)(z-0.3)}}_{\text{minimum phase}} \underbrace{\frac{(z-4)(z+5)}{(4z-1)(5z+1)}}_{\text{allpass}} \qquad \mathrm{ROC}: |z| > 0.5$$

and invert just the minimum phase component, i.e. $H_2(z) = \frac{(z+0.5)(z-0.3)}{(4z-1)(5z+1)}$.

Then $H_1(z)H_2(z) \neq 1$ but rather $H_1(z)H_2(z) = \mathcal{A}(z)$. Hence, the equalizer $H_2(z)$ corrects the magnitude/power distortion, but leaves some residual phase distortion.

Deterministic System Identification

Problem: We wish to determine the impulse response and/or transfer function of a causal LTI unknown system $\mathcal{H}$. It is assumed that we can measure (but do not control) the input $x[n]$ and the output $y[n]$.

Methods:

1. Deconvolution ($x[0] \neq 0$):

$$h[0] = \frac{y[0]}{x[0]} \text{ and } h[n] = \frac{y[n] - \sum_{k=0}^{n-1} h[k]x[n-k]}{x[0]} \text{ for } n \geq 1$$

2. Ratio of z-tranforms.
   - Measure input $x[n]$ and compute its z-transform $X(z)$.
   - Measure output $y[n]$ and compute its z-transform $Y(z)$.
   - $H(z) = Y(z)/X(z)$.
3. Energy density spectrum
   - Measure input $x[n]$ and compute autocorrelation $r_{xx}[\ell] \stackrel{\text{DTFT}}{\longleftrightarrow} S_{xx}(\omega)$.
   - Measure output $y[n]$ and compute cross-correlation $r_{yx}[\ell] \stackrel{\text{DTFT}}{\longleftrightarrow} S_{yx}(\omega)$.
   - $H(\omega) = S_{yx}(\omega)/S_{xx}(\omega)$ for all ω where $S_{xx}(\omega) \neq 0$.

Conclusions

1. This concludes Chapter 7. You are responsible for all of the material in this chapter except Sections 7.8 (Digital Two-Pairs) and 7.9 (Algebraic Stability Test), even if it wasn't covered in lecture.
2. Please read Chapter 8 before the next lecture and have some questions prepared.
3. The next lecture is on Monday 20-Feb-2012 at 6pm. Part of that lecture will be reserved for review.
4. The midterm exam is scheduled for Monday 27-Feb-2012 and is based on Chapters 1-7 of your textbook. No homework will be due on Monday 27-Feb-2012.