

ECE503: Digital Filter Design

Lecture 9

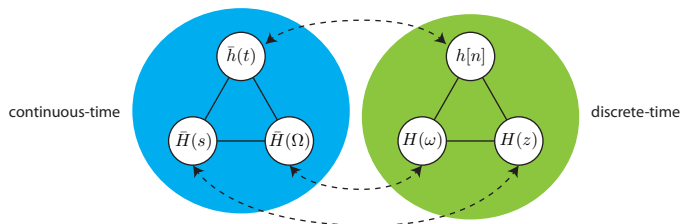
D. Richard Brown III

WPI

26-March-2012

Lecture 9 Topics

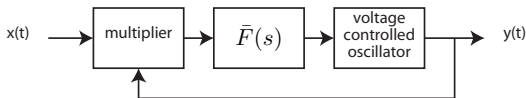
Within the broad topic of **digital filter design**, we are going to focus on the design of LTI IIR discrete-time filters from continuous-time filters.



1. Impulse invariance
2. Bilinear Transform
3. Specification of filter characteristics
4. Frequency transformation of lowpass IIR filters

Motivation/Applications

Application 1: You are given an LTI continuous-time system (block diagram, impulse response, transfer function, ...) and asked to design a discrete-time system that emulates its behavior. How should you do this?



Application 2: You are given a set of discrete-time filter specifications, for example “design a lowpass filter with less than one dB ripple in the passband and at least 30 dB rejection in the stopband, where the passband is $|\omega| < \pi/8$ and the stopband is $|\omega| > \pi/4$ ”, and asked to design a DT filter to meet these specs. How should you do this? One approach:

1. Convert the DT filter specifications to CT filter specifications
2. Design a CT IIR filter to meet the specifications (using long-established relatively simple methods for CT IIR filter design)
3. Transform the CT filter to a DT filter

Why IIR Filters?

Chapter 10 of your textbook covers the design of DT FIR filters (we will not be covering this). Our focus on DT IIR filter design is motivated by the following facts:

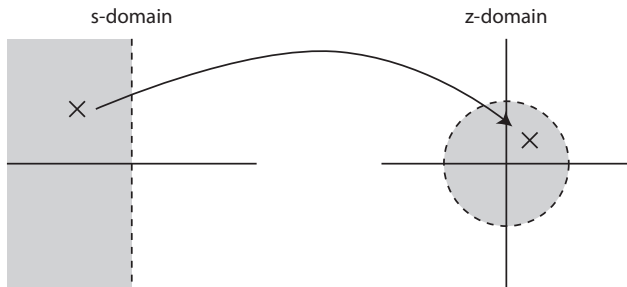
1. All practical LTI CT systems are IIR.
2. To meet a given set of filter specifications, the order of an IIR filter that meets the specifications is typically much lower than the order of an FIR filter that meets the same specifications. Hence the computational complexity of DT IIR filters is typically much less than DT FIR filters that meet the same specifications.

General Requirements for CT/DT Transformation

Requirement 1: Points on the imaginary axis in the s -domain must be mapped to points on the unit circle in the z -domain.

Requirement 2: Stable causal CT transfer functions must be transformed to stable causal DT transfer functions.

Recall that stable causal CT transfer functions have their poles in the left-half plane.



Impulse Invariance

Given a causal LTI CT system impulse response $\bar{h}(t)$, we can write the DT impulse response

$$h[n] = \bar{h}(t)|_{t=nT} = \bar{h}(nT) \text{ for } n = 0, 1, \dots$$

Note: Your textbook defines impulse invariance slightly differently in problem 9.6:

$$h[n] = \begin{cases} \frac{1}{2}\bar{h}(0^+) & n = 0 \\ \bar{h}(nT) & n = 1, 2, \dots \end{cases}$$

The only difference is in $h[0]$. The boxed definition is more common.

Does this CT→DT transformation satisfy the general requirements?

Impulse Invariance Example

Suppose you are given a causal LTI CT system with $\bar{H}(s) = \frac{1}{s-a}$. The inverse Laplace transform gives $\bar{h}(t) = e^{at}\mu(t)$. Hence $h[n] = e^{aTn}\mu[n]$ (or $h[n] = \frac{1}{2}\delta[n] + e^{aTn}\mu[n-1]$ using your textbook's definition).

What is $H(z)$? Rewrite

$$h[n] = e^{aTn}\mu[n] = (e^{aT})^n \mu[n] = \alpha^n \mu[n]$$

Table lookup tells us

$$H(z) = \frac{1}{1 - e^{aT}z^{-1}}$$

with ROC $|z| > e^{aT}$.

Remarks:

- ▶ Suppose a is on the imaginary axis. What is the magnitude of the pole of $H(z)$?
- ▶ Suppose a is complex with negative real part. What can you say about the magnitude of the pole of $H(z)$?

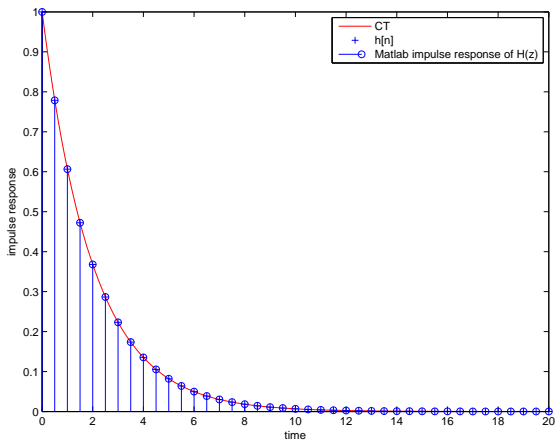
Impulse Invariance Example

```

% CT system
% H(s) = 1/(s-a)
a = -0.5;
tfinal = 20;
ctnum = 1;
ctden = [1 -a];
ctsys = tf(ctnum,ctden);
[y,t] = impulse(ctsys,tfinal);
plot(t,y,'r')

% DT system
T = 0.5;
alpha = exp(a*T);
dtnum = [1 0];
dtdden = [1 -alpha];
dtsys = tf(dtnum,dtdden,T);
[dty,dt] = impulse(dtsys,tfinal);
hold on
n=0:length(dt)-1;
plot(dt,alpha.^n,'b+');
stem(dt,dty);
hold off
xlabel('time');
ylabel('impulse response');
legend('CT','h[n]','Matlab impulse response of H(z)');

```

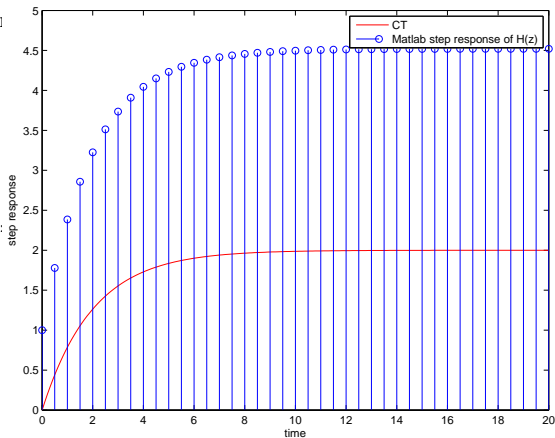


Impulse invariance doesn't imply invariance to all inputs

```

% step responses
% note these results are consistent with
% the final value theorems
[ys,ts] = step(ctsys,tfinal);
[dtys,dtts] = step(dtsys,tfinal);
figure(2)
plot(ts,ys,'r')
hold on
stem(dtts,dtys);
hold off
xlabel('time');
ylabel('step response');
legend('CT','Matlab step response of H(

```



In general, you can pick a particular input and match the response of the DT system to the response of the CT system for that input. But the response to other inputs will be different.

Extending the Previous Impulse Invariance Example

Now suppose you have a causal LTI CT system with transfer function

$$\bar{H}(s) = \sum_{k=1}^N \frac{b_k}{s - a_k}$$

where each a_k is distinct (no repeated roots). Using linearity, we can write

$$\bar{h}(t) = \left(\sum_{k=1}^N b_k e^{a_k t} \right) \mu(t).$$

When we sample this at rate $F_T = \frac{1}{T}$, we get the DT impulse response

$$h[n] = \left(\sum_{k=1}^N b_k e^{a_k T n} \right) \mu[n] = \left(\sum_{k=1}^N b_k (e^{a_k T})^n \right) \mu[n]$$

which has z -transform

$$H(z) = \sum_{k=1}^N \frac{b_k}{1 - e^{a_k T} z^{-1}}$$

with ROC extending outward from the largest magnitude pole.

A Repeated Pole Example

Suppose you have a causal LTI CT system with transfer function

$$\bar{H}(s) = \frac{b_1}{s - a_1} + \frac{b_2}{(s - a_1)^2}.$$

From a Laplace transform table, we have for $\text{Re}(s) > -\alpha$

$$\frac{t^{k-1}}{(k-1)!} e^{-\alpha t} \mu(t) \longleftrightarrow \frac{1}{(s + \alpha)^k}$$

Hence, using linearity, we can write

$$\bar{h}(t) = b_1 e^{a_1 t} \mu(t) + b_2 t e^{a_1 t} \mu(t).$$

When we sample this at rate $F_T = \frac{1}{T}$, we get the DT impulse response

$$h[n] = b_1 e^{a_1 T n} \mu[n] + (b_2 T) n e^{a_1 T n} \mu[n]$$

which has z -transform

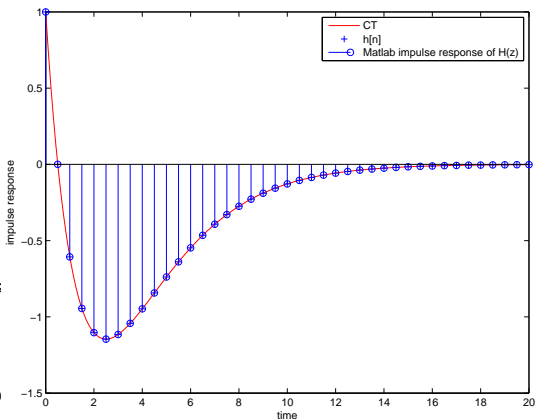
$$H(z) = \frac{b_1}{1 - e^{a_1 T} z^{-1}} + \frac{(b_2 T) e^{a_1 T} z^{-1}}{(1 - e^{a_1 T} z^{-1})^2}$$

with ROC $|z| > |e^{a_1 T}|$.

A Repeated Pole Example

```
% CT system
% H(s) = b1/(s-a1)+b2/(s-a1)^2
a1 = -0.5;
b1 = 1;
b2 = -2;
tfinal = 20;
ctnum = b1*[1 -a1]+[0 b2];
ctden = [1 -2*a1 a1^2];
ctsys = tf(ctnum,ctden);
[y,t] = impulse(ctsys,tfinal);
plot(t,y,'r')
```

```
% DT system
T = 0.5;
alpha = exp(a*T);
dtnum = b1*[1 -alpha 0]+b2*T*alpha*[0 1 0];
dtden = [1 -2*alpha alpha^2];
dtsys = tf(dtnum,dtden,T);
[dtv,dtv] = impulse(dtsys,tfinal);
hold on
n=0:length(dtt)-1;
plot(dtt,b1*alpha.^n+b2*T*n.*alpha.^n,'b+')
stem(dtt,dtv);
hold off
xlabel('time');
ylabel('impulse response');
legend('CT','h[n]','Matlab impulse response of H(z)');
```



Impulse Invariance: Frequency Response

Recall that $h[n]$ is related to $h(t)$ through ideal sampling. What is the relationship between $H(\omega)$ and $\bar{H}(\Omega)$?

We've covered the relationship between the CTFT and the DTFT a few times:

$$H(\omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} \bar{H}\left(\frac{\omega - k2\pi}{T}\right)$$

Remarks:

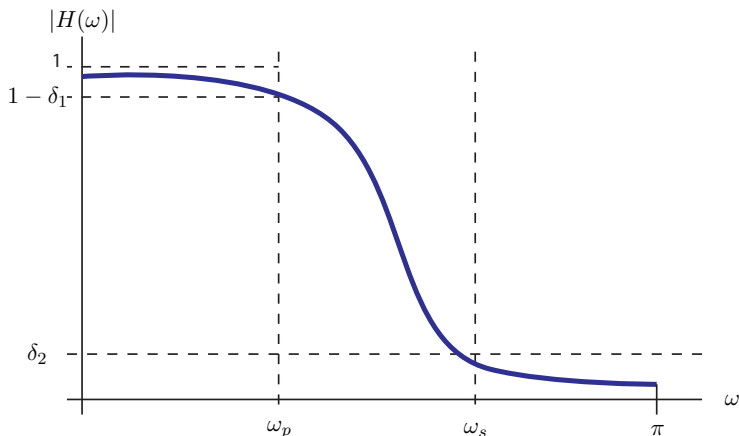
- ▶ If $\bar{H}(\Omega)$ is bandlimited such that $\bar{H}(\Omega) \approx 0$ for all $\Omega \geq \frac{\pi}{T}$, then there will be negligible overlap in the sum of shifted spectra and

$$H(\omega) \approx \frac{1}{T} \bar{H}\left(\frac{\omega}{T}\right) \text{ for } |\omega| \leq \pi$$

Hence, the DT system designed via impulse invariance accurately emulates the CT system's frequency response (and can be scaled if desired).

- ▶ If $\bar{H}(\Omega)$ is not bandlimited, e.g. $\bar{H}(\Omega)$ is a notch filter, there will be aliasing and the resulting DT system's frequency response is not likely to be an accurate emulation of the CT system's frequency response.

Impulse-Invariant Lowpass Butterworth Filter Design Ex.



We start with the desired specifications of the DT filter. For this example, we will use $\omega_p = 0.2\pi$, $\omega_s = 0.3\pi$, $1 - \delta_1 = 0.89125$, and $\delta_2 = 0.17783$.

Impulse-Invariant Lowpass Butterworth Filter Design Ex.

We will use the Butterworth filter approach in this example. Some facts about Butterworth CT lowpass filters (from Appendix A.2):

$$|\bar{H}(\Omega)|^2 = \frac{1}{1 + \left(\frac{\Omega}{\Omega_c}\right)^{2N}}$$

where Ω_c is the cutoff frequency (radians/second) and N is the filter order.

Note:

- ▶ $|\bar{H}(\Omega = 0)|^2 = 1$.
- ▶ $|\bar{H}(\Omega)|^2$ is monotonically decreasing in Ω .

We will use the following steps to design our DT filter:

1. Convert DT filter specs to CT filter specs.
2. Determine the required filter order N and CT cutoff frequency Ω_c .
3. Determine $\bar{H}(s)$.
4. Determine $H(z)$ via impulse invariance.

Impulse-Invariant Lowpass Butterworth Filter Design Ex.

[*** details on board ***]

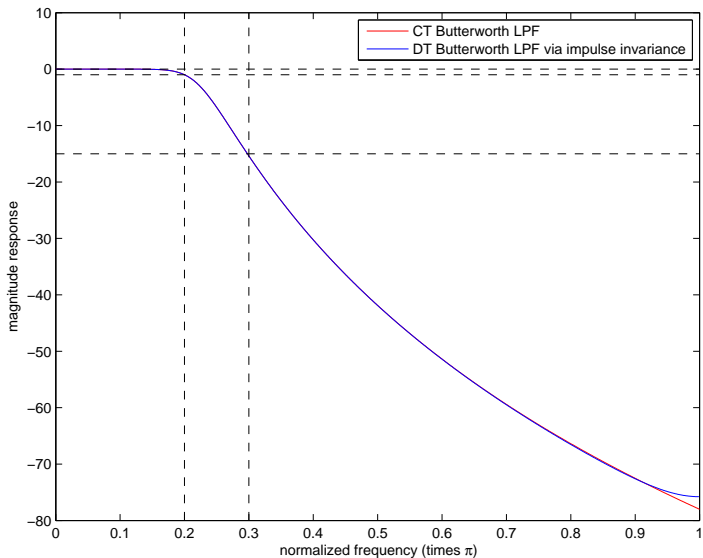
Main result from step 3:

$$\bar{H}(s) = \frac{0.12093}{(s^2 + 0.3640s + 0.4945)(s^2 + 0.9945s + 0.4945)(s^2 + 1.3585s + 0.4945)}$$

Main result from step 4:

$$\begin{aligned} H(z) = & \frac{0.2871 - 0.4466z^{-1}}{1 - 1.2971z^{-1} + 0.6949z^{-2}} \\ & + \frac{-2.1428 + 1.1455z^{-1}}{1 - 1.0691z^{-1} + 0.3699z^{-2}} \\ & + \frac{1.8577 - 0.6303z^{-1}}{1 - 0.9972z^{-1} + 0.2570z^{-2}} \end{aligned}$$

Impulse-Invariant Lowpass Butterworth Filter Design Ex.



Impulse Invariance: Final Remarks

1. Main idea is to preserve characteristics of the CT impulse response in the DT impulse response, e.g. fast settling time, etc.
2. Frequency response of DT system is only a good replica of CT system if the CT system is bandlimited (negligible aliasing).
3. Idea can be extended to other types of waveform invariance, e.g. step invariance.
4. The impulse invariance technique maps poles from $s = a_k$ on the s -plane to $z = e^{a_k T}$ on the z -plane.
5. It is tempting to say that $z = e^{sT}$, which is a one-to-one mapping, hence $s = \ln(z)/T$ and

$$H(z) = \bar{H}(s)|_{s=\ln(z)/T}.$$

This is incorrect in most cases, however.

6. The impulse response of a DT IIR filter is not directly useful for implementation. We need a transfer function or a difference equation.
7. See Matlab function `impinvar`.

Bilinear Transform

Idea: Given a causal stable LTI CT filter $\bar{H}(s)$, we simply substitute

$$s = \frac{2}{T} \left(\frac{1 - z^{-1}}{1 + z^{-1}} \right)$$

to get $H(z)$.

Remark: This substitution is based on converting $\bar{H}(s)$ to a differential equation, performing trapezoidal numerical integration with step size T to get a difference equation, and then converting the difference equation to a transfer function $H(z)$.

One appeal of the bilinear transform is that we have a direct method to go from $\bar{H}(s)$ to $H(z)$ that always works without going through the time-domain.

Your textbook shows:

- ▶ This is a one-to-one mapping between points in the s -plane and z -plane.
- ▶ Points in the left-half (right-half) s -plane are mapped to points inside (outside) the unit circle on the z -plane.
- ▶ There is no aliasing even if $\bar{H}(s)$ is not bandlimited.

Bilinear Transform: Simple Example

Suppose you are given a causal LTI CT system with $\bar{H}(s) = \frac{1}{s-a}$. We can compute $H(z)$ straightforwardly with a little algebra:

$$\begin{aligned}
 H(z) &= \bar{H}(s) \Big|_{s=\frac{2}{T} \left(\frac{1-z^{-1}}{1+z^{-1}} \right)} \\
 &= \frac{1}{\frac{2}{T} \left(\frac{1-z^{-1}}{1+z^{-1}} \right) - a} \\
 &= \frac{T(1+z^{-1})}{2(1-z^{-1}) - aT(1+z^{-1})} \\
 &= \frac{T(1+z^{-1})}{(2-aT) - (2+aT)z^{-1}} \\
 &= \frac{\beta(1+z^{-1})}{1-\alpha z^{-1}}
 \end{aligned}$$

This is quite different than the $H(z)$ we computed via impulse invariance.

Bilinear Transform: Frequency Response

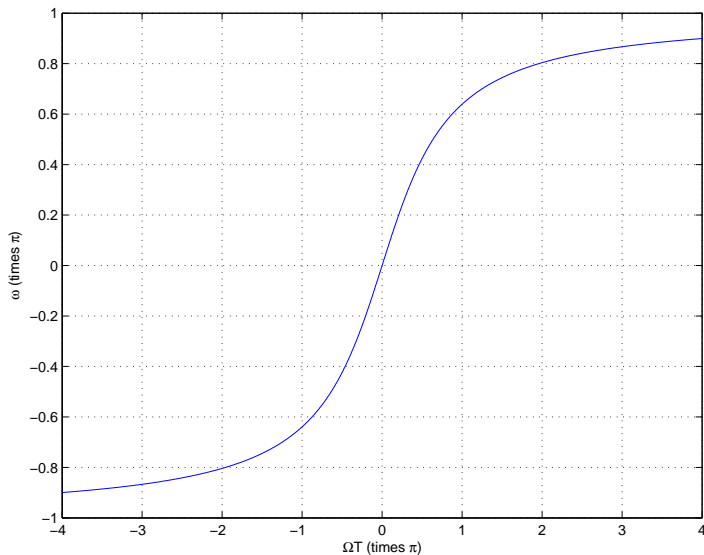
Given a causal stable LTI CT filter $\bar{H}(s)$, we can compute $H(z)$ with via the bilinear transform. What is the relationship between $\bar{H}(\Omega)$ and $H(\omega)$?

Recall that $\bar{H}(\Omega) = \bar{H}(s)|_{s=j\Omega}$ and $H(\omega) = H(z)|_{z=e^{j\omega}}$. Substituting these into the bilinear transform formula, we get

$$\begin{aligned}
 j\Omega &= \frac{2}{T} \left(\frac{1 - e^{-j\omega}}{1 + e^{-j\omega}} \right) \\
 &= \frac{2}{T} \left(\frac{e^{j\omega/2} - e^{-j\omega/2}}{e^{j\omega/2} + e^{-j\omega/2}} \right) \\
 &= \frac{2j}{T} \left(\frac{\frac{1}{2j}(e^{j\omega/2} - e^{-j\omega/2})}{\frac{1}{2}(e^{j\omega/2} + e^{-j\omega/2})} \right) \\
 &= \frac{2j}{T} \tan(\omega/2)
 \end{aligned}$$

Hence $\Omega = \frac{2}{T} \tan(\omega/2)$ or $\omega = 2 \tan^{-1}(\Omega T/2)$.

Bilinear Transform: Frequency Warping



Bilinear Transform: Consequences of Frequency Warping

The good news is that we don't have to worry about aliasing. The bad news is that we have to account for frequency warping when we start from a discrete-time filter specification.

Procedure:

1. Convert the DT filter specifications to CT filter specifications **including prewarping the band edge frequencies**
2. Design a CT IIR filter to meet the specifications
3. Transform the CT filter to a DT filter via the bilinear transform

Bilinear Transform Lowpass Butterworth Filter Design Ex.

As before, we start with the desired specifications of the DT filter. We will use the same parameters as last time: $\omega_p = 0.2\pi$, $\omega_s = 0.3\pi$, $1 - \delta_1 = 0.89125$, and $\delta_2 = 0.17783$.

Steps:

1. Convert DT filter specs to CT filter specs (including pre-warping frequencies).
2. Determine the required filter order N and CT cutoff frequency Ω_c .
3. Determine $\bar{H}(s)$.
4. Determine $H(z)$ via impulse invariance.

Impulse-Invariant Lowpass Butterworth Filter Design Ex.

[*** details on board ***]

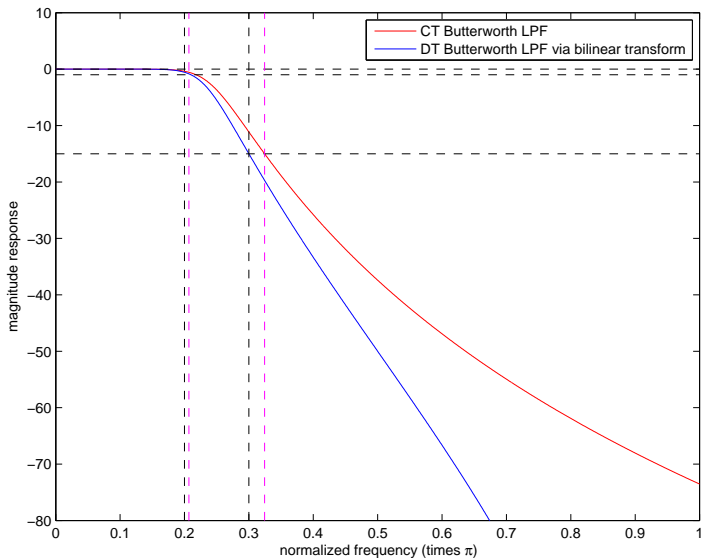
Main result from step 3:

$$\bar{H}(s) = \frac{0.20238}{(s^2 + 0.3966s + 0.5871)(s^2 + 1.0836s + 0.5871)(s^2 + 1.4802s + 0.5871)}$$

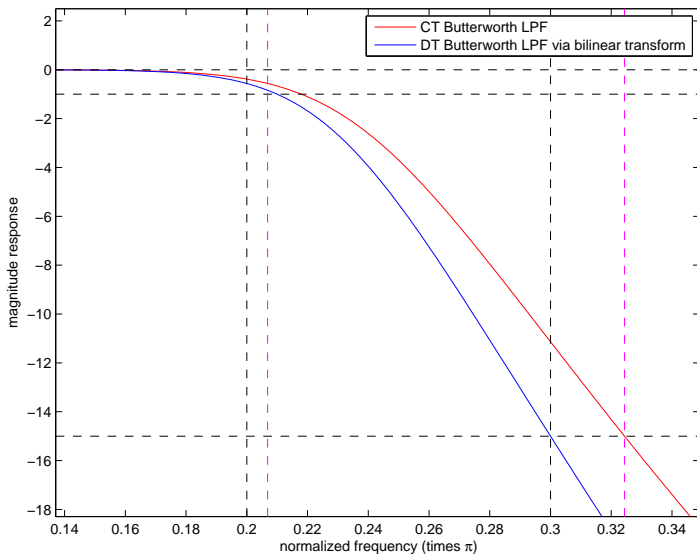
Main result from step 4:

$$H(z) = \frac{0.0007378(1 + z^{-1})^6}{(1 - 1.2686z^{-1} + 0.7051z^{-2})(1 - 1.0106z^{-1} + 0.3583z^{-2})(1 - 0.9044z^{-1} + 0.2155z^{-2})}$$

Bilinear Transform Lowpass Butterworth Filter Design Ex.

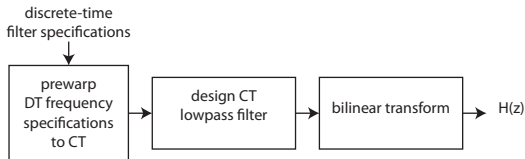


Bilinear Transform Lowpass Butterworth Filter Design Ex.

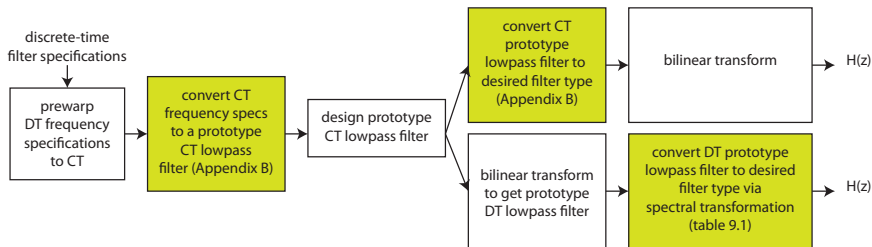


General Filter Design Procedure

The bilinear transform **lowpass filter** design procedure is straightforward:



If you want a different type of filter, e.g. bandpass, there are two options:



Bilinear Transform Bandpass Butterworth Filter Design Ex.

Specifications: $\omega_{p1} = 0.45\pi$, $\omega_{p2} = 0.65\pi$, $\omega_{s1} = 0.3\pi$, $\omega_{s2} = 0.75\pi$,
passband ripple less than 1 dB, minimum stopband attenuation 40 dB.

```
% This code follows example 9.4
% set sampling period
T = 1;

% prewarp frequencies
omega = [0.3 0.45 0.65 0.75]*pi;
Omega_prewarped = (2/T)*tan(omega/2);

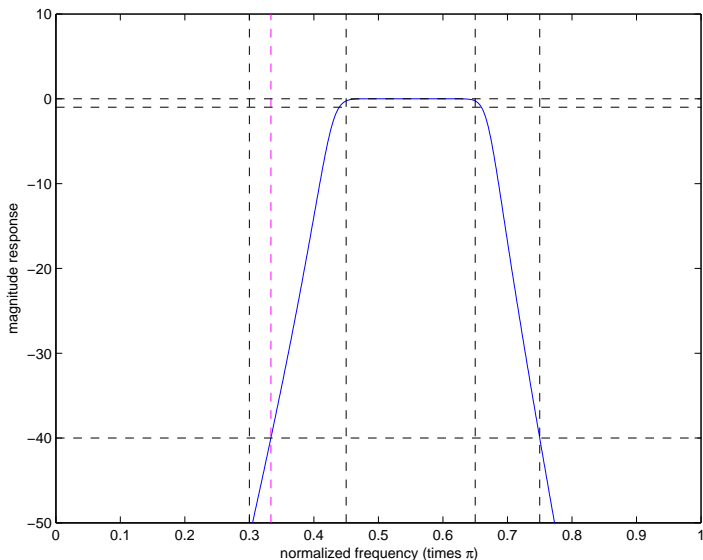
% convert CT frequency specs to a prototype CT lowpass filter
% we want the two stopband frequencies to be GEOMETRICALLY symmetric
% around the GEOMETRIC center frequency of the passband
Omega_0 = sqrt(Omega_prewarped(2)*Omega_prewarped(3));
BW = Omega_prewarped(3)-Omega_prewarped(2);

% adjust lower stopband edge so that product of stopband edge frequencies
% matches Omega_0^2
Omega_prewarped(1) = Omega_0^2/Omega_prewarped(4);

% create prototype CT LPF filter specs (see Appendix B)
Omega_p = 1;
Omega_s = (Omega_0^2-Omega_prewarped(1)^2)/(Omega_prewarped(1)*BW);

% design prototype CT LPF (can also do this as shown in on board)
[N,Wn] = buttord(Omega_p,Omega_s,1,40,'s');
[B,A] = butter(N,Wn,'s');
[BT,AT] = lp2bp(B,A,Omega_0,BW);
[num,den] = bilinear(BT,AT,T);
```

Bilinear Transform Lowpass Butterworth Filter Design Ex.



Bilinear Transform: Final Remarks

1. The bilinear transform only attempts to emulate the magnitude response of the CT filter. The phase response of a DT filter derived via the bilinear transform may be totally different than the CT filter.
2. See section 9.2.2 for several low-order filter examples.
3. See Matlab function `bilinear`.
4. The spectral transformations in Table 9.1 are also handled conveniently with Matlab functions `iirlp2lp`, `iirlp2hp`, `iirlp2bp`, and `iirlp2bs`.
5. Other CT filter design methods can also be used, e.g. Chebychev, elliptic, ...

Conclusions

1. This concludes our coverage of Chapter 9. Our focus was on Sections 9.1-9.6, plus some additional material on impulse invariance. We also referred to material in Appendices A-B of your textbook.
2. We are skipping Chapter 10, but it may be of interest to you if you want to know how to design good FIR digital filters. The techniques in Chapter 10 are **not** based on analog filter design.
3. Please read FFT basics (Section 11.3) and Number Representation (Section 11.8) before the next lecture.
4. The next lecture is on Monday 02-April-2012 at 6pm.
5. The bonus midterm exam (60 points, 60 minutes, based on fundamental material from Chapters 1-7) will begin at 7:30pm on Monday 02-April-2012.