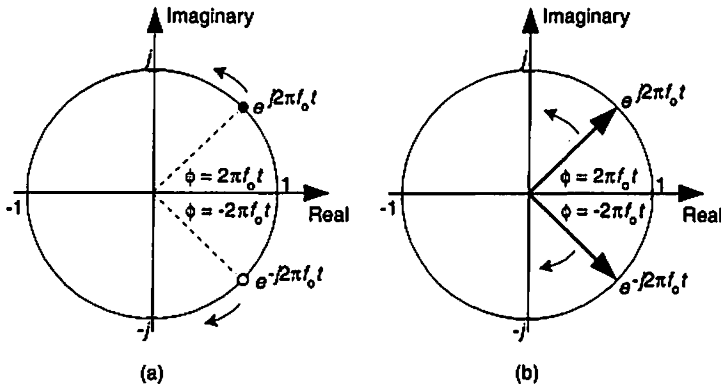

Figure 8-5 A snapshot, in time, of two complex numbers whose exponents change with time: (a) numbers shown as dots; (b) numbers shown as phasors.

Let's pause for a moment here to catch our breath. Don't worry if the ideas of imaginary numbers and the complex plane seem a little mysterious. It's that way for everyone at first—you'll get comfortable with them the more you use them. (Remember, the j-operator puzzled Europe's heavyweight mathematicians for many years.) Granted, not only is the mathematics of complex numbers a bit strange at first, but the terminology is almost bizarre. While the term *imaginary* is an unfortunate one to use, the term *complex* is downright weird. When first encountered, the phrase "complex numbers" makes us think *complicated numbers*. This is regrettable because the concept of complex numbers is not really so complicated.† Just know that the purpose of the above mathematical rigmarole was to validate Eqs. (8–2), (8–3), (8–7), and (8–8). Now, let's (finally!) talk about time-domain signals.

8.3 REPRESENTING REAL SIGNALS USING COMPLEX PHASORS

We now turn our attention to a complex number that is a function of time. Consider a number whose magnitude is one, and whose phase angle increases with time. That complex number is the $e^{j2\pi f_o t}$ point shown in Figure 8–5(a). (Here the $2\pi f_o$ term is frequency in radians/second, and it corresponds to a frequency of f_o cycles/second where f_o is measured in Hz.) As time t gets larger, the complex number's phase angle increases and our number orbits

† The brilliant American engineer Charles P. Steinmetz, who pioneered the use of real and imaginary numbers in electrical circuit analysis in the early twentieth century, refrained from using the term *complex numbers*—he called them *general numbers*.

the origin of the complex plane in a CCW direction. Figure 8–5(a) shows the number, represented by the solid dot, frozen at some arbitrary instant in time. If, say, the frequency $f_o = 2$ Hz, then the dot would rotate around the circle two times per second. We can also think of another complex number $e^{-j2\pi f_o t}$ (the white dot) orbiting in a clockwise direction because its phase angle gets more negative as time increases.

Let's now call our two complex expressions, $e^{j2\pi f_o t}$ and $e^{-j2\pi f_o t}$, quadrature signals. Each has both real and imaginary parts, and they are both functions of time. Those $e^{j2\pi f_o t}$ and $e^{-j2\pi f_o t}$ expressions are often called *complex exponentials* in the literature.

We can also think of those two quadrature signals, $e^{j2\pi f_o t}$ and $e^{-j2\pi f_o t}$, as the tips of two phasors rotating in opposite directions, as shown in Figure 8–5(b). We're going to stick with this phasor notation for now because it'll allow us to achieve our goal of representing real sinusoids in the context of the complex plane. Don't touch that dial!

To ensure that we understand the behavior of a simple quadrature signal, Figure 8–6 shows the three-dimensional path of the $e^{j2\pi f_o t}$ signal as time passes. We've added the time axis, coming out of the page, to show how $e^{j2\pi f_o t}$ follows a corkscrew path spiraling along, and centered about, the time axis. The real and imaginary parts of $e^{j2\pi f_o t}$ are shown as the sine and cosine projections in Figure 8–6 and give us additional insight into Eq. 8–7.

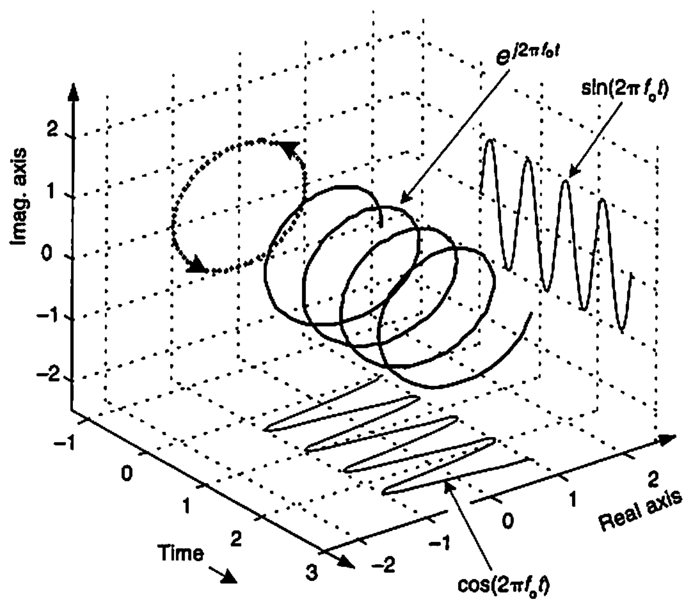

Figure 8–6 The motion of the $e^{j2\pi f_o t}$ complex signal as time increases.

To appreciate the physical meaning of our discussion here, let's remember that a continuous quadrature signal $e^{j2\pi f_0 t} = \cos(2\pi f_0 t) + j\sin(2\pi f_0 t)$ is not just mathematical mumbo jumbo. We can generate $e^{j2\pi f_0 t}$ in our laboratory and transmit it to another lab down the hall. All we need is two sinusoidal signal generators, set to the same frequency f_0. (However, somehow we have to synchronize those two hardware generators so their relative phase shift is fixed at 90 degrees.) Next we connect coax cables to the generators' output connectors and run those two cables, labeled *cos* for the cosine signal and *sin* for the sinewave signal, to their destination as shown in Figure 8–7.

Now for a two-question pop quiz. First question: In the other lab, what would we see on the screen of an oscilloscope if the continuous real $\cos(2\pi f_0 t)$ and $\sin(2\pi f_0 t)$ signals were connected to the horizontal and vertical input channels, respectively, of the scope (remembering, of course, to set the scope's horizontal sweep control to the External position)? That's right. We'd see the scope's electron beam rotating counterclockwise in a circle on the scope's screen.

Next, what would be seen on the scope's display if the cables were mislabeled and the two signals were inadvertently swapped? We'd see another circle, but this time it would be orbiting in a clockwise direction. This would be a neat little real-world demonstration if we set the signal generators' f_0 frequencies to, say, 1 Hz.

This oscilloscope example is meaningful and helps us answer the important question "When we work with quadrature signals, how is the j-operator implemented in hardware?" The j-operator is implemented by how we treat the two signals relative to each other. We have to treat them orthogonally such that the real $\cos(2\pi f_0 t)$ signal represents an east-west value, and the real $\sin(2\pi f_0 t)$ signal represents an orthogonal north-south value. (By "orthogonal," I mean the north-south direction is oriented exactly 90 degrees relative to the east-west direction.) So in our oscilloscope example the j-operator is implemented merely by how the connections are made to the scope. The real cosine signal controls horizontal deflection and the real sine signal controls vertical deflection. The result is a two-dimensional quadrature signal represented by the instantaneous position of the dot on the scope's display. We physically implemented the j-operator in $e^{j2\pi f_0 t}=\cos(2\pi f_0 t)+j\sin(2\pi f_0 t)$ the moment we connected the $\sin(2\pi f_0 t)$ signal to the vertical input connector of the oscilloscope. Our Figure 8–7 example reminds us of an important characteris-

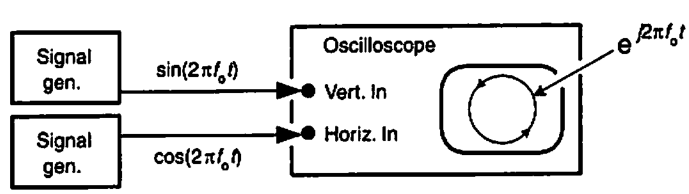

Figure 8–7 Displaying a quadrature signal using an oscilloscope.

tic of quadrature signals: While real signals can be transmitted over a single cable, two cables are always necessary to transmit a quadrature (complex) signal.

Returning to Figure 8–5(b), ask yourself: "What's the vector sum of those two phasors as they rotate in opposite directions?" Think about this for a moment. That's right, the phasors' real parts will always add constructively, and their imaginary parts will always cancel. This means the summation of these $e^{j2\pi f_o t}$ and $e^{-j2\pi f_o t}$ phasors will always be a purely real number. Implementations of modern-day digital communications systems are based on this property!

To emphasize the importance of the real sum of these two complex sinusoids we'll draw yet another picture. Consider the waveform in the three-dimensional Figure 8–8 generated by the sum of two half-magnitude complex phasors, $e^{j2\pi f_o t}/2$ and $e^{-j2\pi f_o t}/2$, rotating in opposite directions about, and moving down along, the time axis.

Thinking about these phasors, it's clear now why the cosine wave can be equated to the sum of two complex exponentials by

$$\cos(2\pi f_o t) = \frac{e^{j2\pi f_o t} + e^{-j2\pi f_o t}}{2} = \frac{e^{j2\pi f_o t}}{2} + \frac{e^{-j2\pi f_o t}}{2}. \qquad (8\text{–}13)$$

Eq. (8–13), a well-known and important expression, is also one of Euler's identities. We could have derived this identity by solving Eqs. (8–7) and (8–8) for $j\sin(\varnothing)$, equating those two expressions, and solving that final equation for $\cos(\varnothing)$. Similarly, we could go through the same algebra exercise and show a real sinewave as also the sum of two complex exponentials as

$$\sin(2\pi f_o t) = \frac{e^{j2\pi f_o t} - e^{-j2\pi f_o t}}{2j} = \frac{je^{-j2\pi f_o t}}{2} - \frac{je^{j2\pi f_o t}}{2}. \qquad (8\text{–}14)$$

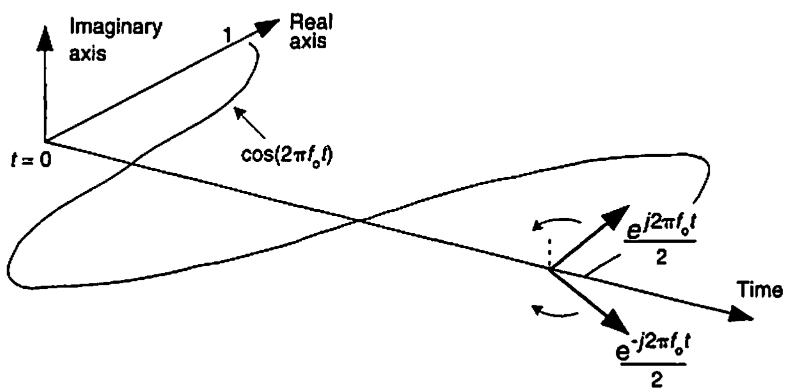

Figure 8–8 A cosine represented by the sum of two rotating complex phasors.

Look at Eqs. (8–13) and (8–14) carefully—they are the standard expressions for a cosine wave and a sinewave, using complex notation, and are seen throughout the literature of quadrature communications systems. Equation (8–13) tells us that the two complex exponentials are both oriented toward the positive real axis when time t=0. The j operators in Eq. (8–14) tell us that the negative-frequency complex exponential is oriented along the positive imaginary axis, and the positive-frequency complex exponential is oriented along the negative imaginary axis, when time t=0.

To keep the reader's mind from spinning like those complex phasors, please realize that the sole purpose of Figures 8–5 through 8–8 is to validate the complex expressions of the cosine and sinewave given in Eqs. (8–13) and (8–14). Those two equations, along with Eqs. (8–7) and (8–8), are the Rosetta Stone of quadrature signal processing.† We can now easily translate, back and forth, between real sinusoids and complex exponentials.

Let's step back now and remind ourselves what we're doing. We are learning how real signals that can be transmitted down a coax cable, or digitized and stored in a computer's memory, can be represented in complex number notation. Yes, the constituent parts of a complex number are each real, but we're treating those parts in a special way—we're treating them in quadrature.

8.4 A FEW THOUGHTS ON NEGATIVE FREQUENCY

It's important for us to be comfortable with the concept of negative frequency because it's essential in understanding the spectral replication effects of periodic sampling, discrete Fourier transforms, and the various quadrature signal processing techniques discussed in Chapter 9. The convention of negative frequency serves as both a consistent and powerful mathematical tool in our analysis of signals. In fact, the use of negative frequency is mandatory when we represent *real signals*, such as a sine or cosine wave, in complex notation.

The difficulty in grasping the idea of negative frequency may be, for some, similar to the consternation felt in the parlors of mathematicians in the Middle Ages when they first encountered negative numbers. Until the thirteenth century, negative numbers were considered *fictitious* because numbers were normally used for counting and measuring. So up to that time, negative numbers just didn't make sense. In those days, it was valid to ask, "How can you hold in your hand something that is less than nothing?" The idea of subtracting six from four must have seemed meaningless. Math historians suggest that negative numbers were first analyzed in Italy. As the story goes, around the year 1200 the Italian mathematician Leonardo da Pisa (known as Fibonacci)

† The Rosetta Stone was a basalt slab found in Egypt in 1799. It had the same text written in three languages, two of them being Greek and Egyptian hieroglyphs. This enabled scholars to, finally, translate the ancient hieroglyphs.

was working on a financial problem whose only valid solution involved a negative number. Undaunted, Leo wrote, "This problem, I have shown to be insoluble unless it is conceded that the first man had a debt." Thus negative numbers arrived on the mathematics scene, never again to be disregarded.

Modern men and women can now appreciate that negative numbers have a direction associated with them. The direction is backward from zero in the context that positive numbers point forward from zero. For example, negative numbers can represent temperatures measured in degrees below zero, minutes before the present if the present is considered as zero time, or money we owe the tax collector when our income is considered positive dollars. So, the notion of negative quantities is perfectly valid if we just define it properly. As comfortable as we now are with negative numbers, negative frequency remains a troublesome and controversial concept for many engineers[3,4]. This author once encountered a paper in a technical journal which stated: "since negative frequencies cannot exist—." Well, like negative numbers, negative frequency is a perfectly valid concept as long as we define it properly relative to what we're used to thinking of as positive frequency. With this thought in mind, we'll call Figure 8–5's $e^{j2\pi f_o t}$ signal a *positive-frequency* complex exponential because it rotates around the complex plane's origin in a circle in a positive-angle direction at a cyclic frequency of f_o cycles per second. Likewise, we'll refer to the $e^{-j2\pi f_o t}$ signal as a *negative-frequency* complex exponential because of its negative-angle direction of rotation.

So we've defined negative frequency in the frequency domain. If my DSP pals want to claim negative frequency doesn't exist in the time domain, I won't argue. However, our frequency-domain negative frequency definition is clean, consistent with real signals, very useful, and here to stay.

8.5 QUADRATURE SIGNALS IN THE FREQUENCY DOMAIN

Now that we know much about the time-domain nature of quadrature signals, we're ready to look at their frequency-domain descriptions. We'll illustrate the full three-dimensional aspects of the frequency domain so none of the phase relationships of our quadrature signals will be hidden from view. Figure 8–9 tells us the rules for representing complex exponentials in the frequency domain.

We'll represent a single complex exponential as a narrow impulse located at the frequency specified in the exponent. In addition, we'll show the phase relationships between those complex exponentials along the real and imaginary frequency-domain axes. To illustrate those phase relationships, a complex frequency domain representation is necessary. With all this said, take a look at Figure 8–10.

See how a real cosine wave and a real sinewave are depicted in our complex frequency-domain representation on the right side of Figure 8–10. Those bold arrows on the right of Figure 8–10 are not rotating phasors but