ECE503 Final Exam

Your Name: SOLUTION Your box #: ____________

April 30, 2012

Tips:

- Look over all of the questions before starting.
- Budget your time to allow yourself enough time to work on each question.
- Write neatly and show your work! Points will be deducted for a disorderly presentation of your solution.
- This exam is worth a total of 500 points.
- This exam is to be completed in 180 minutes.
- You are permitted to consult your textbook, two handwritten "cheat sheets", and a calculator.
- Attach your "cheat sheets" to the exam when you hand it in.

problem 1	problem 2	problem 3	problem 4	total final exam score
/100 points	/120 points	/130 points	/150 points	/500 points

1. 100 points. Suppose you have a discrete-time system with impulse response

$$h[n] = 0.9^n \mu[n]$$

where $\mu[n]$ is the discrete-time unit step function. If an input

$$x[n] = \cos\left(\frac{\pi}{6}n\right)$$

is applied to this system, determine the output $y[n]$.

$$H(z) = \frac{1}{1-0.9z^{-1}} \quad \text{ROC: } |z| > 0.9$$

causal & stable $\Rightarrow$ $H(\omega)$ exists

$$H(\omega) = H(z)\Big|_{z=e^{j\omega}} = \frac{1}{1-0.9e^{-j\omega}}$$

Given an input $x[n] = \cos\left(\frac{\pi}{6}n\right)$, the output

$y[n] = a\cos\left(\frac{\pi}{6}n + \theta\right)$

where a is the magnitude response of $H(\omega = \frac{\pi}{6})$

and θ is the phase response of $H(\omega = \frac{\pi}{6})$.

$$H\left(\omega = \frac{\pi}{6}\right) = \frac{1}{1-0.9e^{-j\frac{\pi}{6}}} = 0.8783 - j1.7917$$

$$= \underbrace{1.9954}_{a} \ \measuredangle \ \underbrace{-1.1150}_{\theta}$$

Hence $\boxed{y[n] = 1.9954\cos\left(\frac{\pi}{6}n - 1.1150\right)}$

2. 120 points total. An LTI system has a transfer function $H(z)$ given by

$$H(z) = \frac{3 + 0.2z^{-1}}{(1 - 0.7z^{-1})(1 + 1.6z^{-1})}.$$

(a) 30 points. List all of the possible regions of convergence for this transfer function.

ROC1 : $|z| < 0.7$

ROC2 : $0.7 < |z| < 1.6$

ROC3 : $1.6 < |z|$

(b) 30 points. Can this transfer function to be stable? Explain.

Yes. $H(z)$ with ROC2 is stable because the ROC contains the unit circle in the complex plane.

(c) 60 points. Determine the impulse response of this system assuming a region of convergence $|z| > 1.6$. Is this system causal? Is this system stable?

$$H(z) = \frac{a}{1-0.7z^{-1}} + \frac{b}{1+1.6z^{-1}}$$

$$\left.\begin{aligned} a+b &= 3 \\ 1.6a - 0.7b &= 0.2 \end{aligned}\right\} \Rightarrow \begin{aligned} a &= 1 \\ b &= 2 \end{aligned}$$

Hence $\boxed{h[n] = (0.7)^n u[n] + 2(-1.6)^n u[n]}$

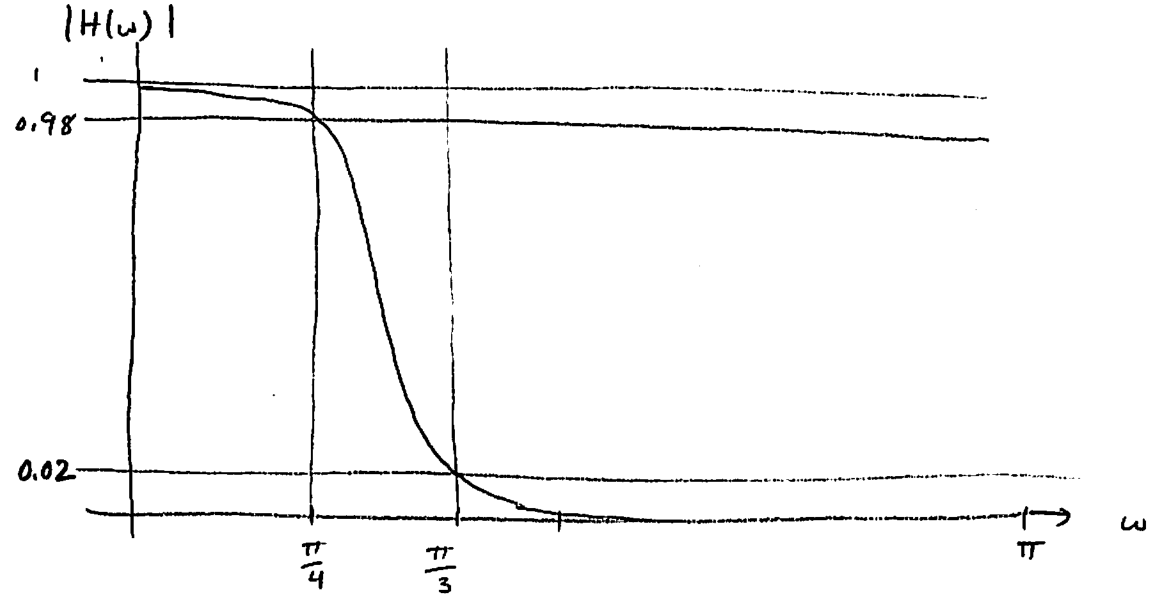
causal but not stable

We know causality from the fact that the ROC extends outward from the largest pole.

3. 130 points total. Suppose you need to design a linear time-invariant discrete-time lowpass filter that satisfies the following specifications:

$$0.98 \leq |H(\omega)| \leq 1 \text{ for } 0 \leq \omega \leq \frac{\pi}{4}$$
$$|H(\omega)| \leq 0.02 \text{ for } \frac{\pi}{3} \leq \omega \leq \pi$$

(a) 30 points. Would the impulse invariance filter design method be appropriate here? Explain.



This is clearly a lowpass filter with very little high frequency content. Since we dont need to worry about aliasing, the impulse invariance method would be fine here.

(b) 100 points. Suppose you decide to use the Butterworth lowpass filter design technique to compute an appropriate analog filter and the bilinear transform to convert this analog filter to a discrete time filter. Recall that Butterworth lowpass filters have a magnitude squared response

$$|\bar{H}(\Omega)|^2 = \frac{1}{1+\left(\frac{\Omega}{\Omega_c}\right)^{2N}}$$

where Ω_c is the 3dB cutoff frequency and N is the filter order. Determine Ω_c and N such that the discrete-time specifications will be satisfied after the bilinear transform is applied $\bar{H}(s) \to H(z)$. Note: You do not need to compute $\bar{H}(s)$ or perform the bilinear transform in this problem. You just need to determine Ω_c and N for the analog filter. Show your work.

step 1: pre-warp band edges because we are using the bilinear transform

$\Omega_0 = \frac{2}{T}\tan\left(\frac{\omega}{2}\right)$ T doesn't matter, so we'll pick $T = 1$

$\Rightarrow$ CT filter specs : $0.98 \le |\bar{H}(\Omega)| \le 1$ for $0 \le \Omega \le 0.8284$

$\bar{H}(\Omega) \le 0.02$ for $\Omega \ge 1.1547$

step 2: solve for Ω_c and N.

use the fact that the butterworth magnitude response is monotonically decreasing to set up two equations and two unknowns

$$\frac{1}{1+\left(\frac{0.8284}{\Omega_c}\right)^{2N}} = (0.98)^2 \qquad (i)$$

$$\frac{1}{1+\left(\frac{1.1547}{\Omega_c}\right)^{2N}} = (0.02)^2 \qquad (ii)$$

continued...

Extra page for calculations (if needed)...

$$(i) \Rightarrow \left(\frac{0.8284}{\Omega_c}\right)^{2N} = \frac{1}{(0.98)^2} - 1 = a \qquad (iii)$$

$$(ii) \Rightarrow \left(\frac{1.1547}{\Omega_c}\right)^{2N} = \frac{1}{(0.02)^2} - 1 = b \qquad (iv)$$

$$(iii) \Rightarrow 2N\left(\log(0.8284) - \log(\Omega_c)\right) = \log a \qquad (v)$$

$$(iv) \Rightarrow 2N\left(\log(1.1547) - \log(\Omega_c)\right) = \log b \qquad (vi)$$

$$2N\left(\log(0.8284) - \log(1.1547)\right) = \log(a) - \log(b)$$

$$N = \frac{\log(a) - \log(b)}{2\left(\log(0.8284) - \log(1.1547)\right)} = 16.58$$

hence we should select $\boxed{N = 17}$

If we match the passband specs exactly, we have

$$2 \cdot 17\left(\log(0.8284) - \log(\Omega_c)\right) = \log a$$

$$\log(\Omega_c) = \log(0.8284) - \frac{\log a}{34} =$$

$$\Rightarrow \boxed{\Omega_c = 0.9098} \quad \text{(passband exact)}$$

If we match the stopband specs exactly, we have

$$2 \cdot 17\left(\log(1.1547) - \log(\Omega_c)\right) = \log b$$

$$\log(\Omega_c) = \log(1.1547) - \frac{\log b}{34} =$$

$$\Rightarrow \boxed{\Omega_c = 0.9173} \quad \text{(stopband exact)}$$

4. 150 points total. Suppose you are given a linear time invariant IIR filter with the transfer function

$$\begin{aligned} H(z) &= \frac{Y(z)}{X(z)} \\ &= \frac{0.6(1-z^{-1})^2(1-2z^{-1})^2}{(1-(0.7+0.7j)z^{-1})(1-(0.7-0.7j)z^{-1})(1-(0.4+0.9j)z^{-1})(1-(0.4-0.9j)z^{-1})} \\ &= \frac{0.6-3.6z^{-1}+7.8z^{-2}-7.2z^{-3}+2.4z^{-4}}{1-2.2z^{-1}+3.07z^{-2}-2.142z^{-3}+0.9506z^{-4}} \end{aligned}$$

Note that $j=\sqrt{-1}$.

(a) 50 points. Suppose you wish to implement this filter as a **direct form I** filter with <u>all</u> filter coefficients stored as **signed 8-bit fixed-point numbers.** How many fractional bits should your filter coefficients have to minimize the coefficient quantization error? Explain.

We see the largest filter coefficient is 7.8

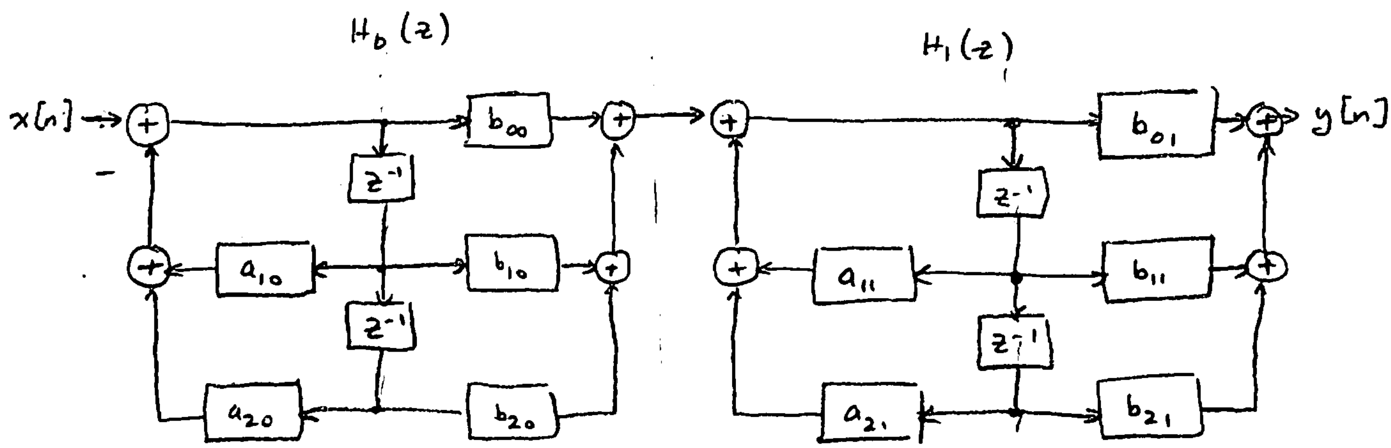
This will require 3 non fractional bits to avoid overflow.

we also need a sign bit.

Hence <u>4 bits</u> remain that we can use as fractional bits. This is the best choice because more fractional bits result in overflow when we quantize the coefficients. Less fractional bits provides less precision.

Continued...

(b) 100 points. Draw a "Direct Form II – Second Order Sections" realization of $H(z)$ assuming **infinite precision filter coefficients** (each second order section should have *real-valued* coefficients). Draw neatly and label everything accurately for full credit.



To calculate the coefficients, we can go with (as one possibility)

$$H_0(z) = \frac{0.6\,(1-z^{-1})^2}{(1-(0.7+0.7j)z^{-1})(1-(0.7-0.7j)z^{-1})}$$

$$= \frac{0.6\,(1-2z^{-1}+z^{-2})}{1-1.4z^{-1}+0.98} = \frac{0.6-1.2z^{-1}+0.6z^{-2}}{1-1.4z^{-1}+0.98}$$

$$\boxed{\begin{array}{l} b_{00} = 0.6\ ;\ b_{10} = -1.2\ ;\ b_{20} = 0.6 \\ a_{10} = -1.4\ ;\ a_{20} = 0.98 \end{array}}$$

$$H_1(z) = \frac{(1-2z^{-1})^2}{(1-(0.4+0.9j)z^{-1})(1-(0.4-0.9j)z^{-1})}$$

$$= \frac{1-4z^{-1}+4z^{-2}}{1-0.8z^{-1}+0.97}$$

$$\boxed{\begin{array}{l} b_{01} = 1\ ;\ b_{11} = -4\ ;\ b_{21} = 4 \\ a_{11} = -0.8\ ;\ a_{21} = 0.97 \end{array}}$$