1. 5 points total.

(a) 3 points. Analytically confirm the result in equation (12.47), i.e. calculate the matrix/vector products and apply any useful trigonometric identities to arrive at the final result in Example 12.2.

**Solution:** We just do a little linear algebra and apply the necessary trig identities as follows:

\[
\begin{bmatrix}
\Delta r \\
\Delta \theta
\end{bmatrix} = \begin{bmatrix}
\frac{1}{2r} & 0 \\
\frac{1}{2r^2 \tan \theta} & -\frac{1}{2r \sin \theta}
\end{bmatrix} \begin{bmatrix}
2r \cos \theta & 2r \sin \theta
\end{bmatrix} \begin{bmatrix}
\Delta \alpha \\
\Delta \beta
\end{bmatrix}
\]

\[
= \begin{bmatrix}
\cos \theta & \sin \theta \\
\frac{1}{r} \cos \theta - \frac{1}{r \sin \theta} & \frac{1}{r} \sin \theta
\end{bmatrix} \begin{bmatrix}
\Delta \alpha \\
\Delta \beta
\end{bmatrix}
\]

\[
= \begin{bmatrix}
\cos \theta & \sin \theta \\
\frac{1}{r} \cos^2 \theta - \frac{1}{r \sin \theta} & \frac{1}{r} \sin \theta
\end{bmatrix} \begin{bmatrix}
\Delta \alpha \\
\Delta \beta
\end{bmatrix}
\]

\[
= \begin{bmatrix}
\cos \theta & \sin \theta \\
\frac{1}{r} \sin^2 \theta - \frac{1}{r \sin \theta} & \frac{1}{r} \cos \theta
\end{bmatrix} \begin{bmatrix}
\Delta \alpha \\
\Delta \beta
\end{bmatrix}
\]

\[
= \begin{bmatrix}
\cos \theta & \sin \theta \\
\frac{1}{r} \sin \theta & \frac{1}{r} \cos \theta
\end{bmatrix} \begin{bmatrix}
\Delta \alpha \\
\Delta \beta
\end{bmatrix}
\]

This is the desired result.

(b) 2 points. Numerically verify the result for a particular choice of pole locations \( r = 0.9, \theta = \pi/4 \), and coefficient quantization errors \( \Delta \alpha = 0.01 \) and \( \Delta \beta = -0.01 \). In other words, numerically compute the exact radial and angular displacement of the poles from the transfer function with quantized and unquantized \( \alpha \) and \( \beta \) and then use equation (12.47) to compute the analytically predicted approximate displacement of the poles. Is the analytical prediction accurate?

**Solution:** Here is my code.

% ECE503 Spring 2012

% original unquantized poles, coupled form parameters (see p.674)
\( r = 0.9; \)
\( \theta = \pi/4; \)
\( \alpha = r \cos(\theta); \)
\( \beta = r \sin(\theta); \)
\( \lambda = \text{roots}([1 -2*\alpha (\alpha^2+\beta^2)]); \) % unquantized roots

% quantized roots
delta_alpha = 0.01;
delta_beta = -0.01;
alphahat = alpha + delta_alpha;
betahat = beta + delta_beta;
lamhat = roots([1 -2*alphahat (alphahat^2+betahat^2)]); % exact quantized roots
rhat = abs(lamhat);
theta = angle(lamhat);
delta_r_exact = rhat - [r;r]
delta_theta_exact = theta - [theta;-theta]

% theoretical prediction via approximation
delta_r_approx = cos(theta)*delta_alpha + sin(theta)*delta_beta
delta_theta_approx = -(1/r)*sin(theta)*delta_alpha + (1/r)*cos(theta)*delta_beta

Here are my results.

delta_r_exact =

1.0e-03 *

0.1111
0.1111

delta_theta_exact =

-0.0157
0.0157

delta_r_approx =

1.7347e-18

delta_theta_approx =

-0.0157

The approximation says the roots should not change radially (the exact solution shows that the radius of the roots changes very slightly, increasing by about 10^{-4}). The approximation also very accurately predicts the angular change of the roots caused by the slight changes in the coupled form parameters $\alpha$ and $\beta$. The approximation says that the angle of the first root will change by about $-0.0157$ (which it does). Since the roots have to appear as complex conjugates, the angle of the second root changes by $+0.0157$. Hence, the analytical prediction is quite accurate in this case.
2. 4 points. Mitra 12.2

**Solution:** Since we are concerned with pole sensitivity here, note that the denominator of both the highpass and lowpass filters in Figure 8.34 is just \( B(z) = z - \alpha \). Since the coefficient \( \alpha \) is real, there is only one pole at \( z = \alpha = r e^{j\theta} \) with \( r = |\alpha| \) and \( \theta = 0 \) if \( \alpha > 0 \) or \( \theta = \pi \) if \( \alpha < 0 \).

When \( \alpha \) is quantized to \( \hat{\alpha} \), we have \( \hat{r} = |\hat{\alpha}| \) and \( \hat{\theta} = 0 \) if \( \hat{\alpha} > 0 \) or \( \hat{\theta} = \pi \) if \( \hat{\alpha} < 0 \).

As long as the quantization of \( \alpha \) is such that \( \hat{\alpha} \) has the same sign as \( \alpha \) (which will be the case for any reasonable quantization scheme), we can say that \( \Delta \theta = 0 \) and

\[
\Delta r = \begin{cases} 
\Delta \alpha & \alpha > 0 \\
-\Delta \alpha & \alpha < 0.
\end{cases}
\]

3. 6 points. Mitra 12.6

**Solution to part (a):**

(a) For direct form implementation \( B(z) = (z - z_1)(z - z_2)(z - z_3) \), where

\[
z_1 = r_1 e^{j\theta_1}, \quad z_2 = r_2 e^{j\theta_2}, \quad \text{and} \quad z_3 = r_3 e^{j\theta_3}.
\]

Thus, \( B(z) = (z^2 - 2n \cos \theta_1 z + n^2)(z - r_3) \).

\[
(z^2 - 0.5z + 0.25)(z + 0.75). \quad \text{This implies,} \quad 2n \cos \theta_1 = 0.5, \quad n^2 = 0.25, \quad r_3 = -0.75, \quad \text{and} \quad \theta_3 = \pi.
\]

Thus, \( n = \sqrt{0.25} = 0.5 \) and \( \cos \theta_1 = \frac{0.5}{2 \times 0.5} = 0.5 \). Now, \( \frac{1}{B(z)} = \frac{\frac{1}{(z^2 - 0.5z + 0.25)(z + 0.75)}}{z - 0.25 + j0.433 + z - 0.25 - j0.433 + z + 0.75} \).

\[
P_1 = \begin{bmatrix} \cos \theta_1 & n & n^2 \cos \theta_1 \end{bmatrix} = \begin{bmatrix} 0.5 & 0.5 & 0.125 \end{bmatrix},
\]

\[
Q_1 = \begin{bmatrix} \sin \theta_1 & 0 & n^2 \sin \theta_1 \end{bmatrix} = \begin{bmatrix} 0.866 & 0 & 0.2165 \end{bmatrix}, \quad R_1 = -0.4211, \quad \text{and} \quad X_1 = 0.972. \quad \text{Likewise},
\]

\[
P_3 = \begin{bmatrix} \cos \theta_3 & r_3 & r_3^2 \cos \theta_3 \end{bmatrix} = \begin{bmatrix} -1 & 0.75 & 0.5625 \end{bmatrix},
\]

\[
Q_3 = \begin{bmatrix} \sin \theta_3 & 0 & r_3^2 \sin \theta_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}, \quad R_3 = 0.8421, \quad \text{and} \quad X_3 = 0.75.
\]

Thus, \( \Delta r_1 = (-R_1 P_1 + X_1 Q_1) \cdot \Delta B = 1.0523 \Delta b_0 + 0.2105 \Delta b_1 + 0.2631 \Delta b_2 \),

\[\Delta \theta_1 = -\frac{1}{n} (X_1 P_1 + R_1 Q_1) \cdot \Delta B = -0.5509 \Delta b_0 - 0.5509 \Delta b_1 - 0.1377 \Delta b_2,
\]

\[\Delta r_3 = (-R_3 P_3 + X_3 Q_3) \cdot \Delta B = 0.8421 \Delta b_0 - 0.6316 \Delta b_1 + 0.4737 \Delta b_2, \quad \text{and}
\]

\[\Delta \theta_3 = -\frac{1}{r_3^2} (X_3 P_3 + R_3 Q_3) \cdot \Delta B = 0.
\]

**Solution to part (b):**

(b) Cascade form: \( B(z) = (z^2 + c_1 z + c_0)(z + d_0) - B_1(z)B_2(z) \), where

\[
B_1(z) = z^2 + c_1 z + c_0 = z^2 - 0.5z + 0.25 = (z - r_1 e^{j\theta_1})(z - r_1 e^{-j\theta_1}) = z^2 - 2r_1 \cos \theta_1 z + r_1^2
\]

and \( B_2(z) = z + d_0 = z - 0.75 = z - r_3 e^{j\theta_3} \). Comparing we get \( 2r_1 \cos \theta_1 = 0.5, \quad r_1^2 = 0.25, \)
\[ r_3 = 0.75, \quad \theta_3 = \pi. \] Solving the first two equations we get \( \eta_1 = \sqrt{0.25} = 0.5 \) and 
\[ \cos \theta_1 = \frac{0.5}{2\sqrt{0.25}} = 0.5. \]

Now, 
\[ B(z) = \frac{-j1.1547}{z - 0.25 - j0.433} + \frac{j1.1547}{z - 0.25 + j0.433}. \]
Hence, \( R_1 = 0 \) and \( X_1 = -1.1547. \)

\[ P_1 = [\cos \theta_1, \eta_1] = [0.5, 0.5], Q_1 = [-\sin \theta_1, 0] = [0.866, 0]. \]

Next, 
\[ \frac{1}{B_2(z)} = \frac{1}{z + 0.75}. \]
Hence, \( R_3 = 1 \) and \( X_3 = 0. \) Here, \( P_3 = \cos \theta_3 = -1, \) and \( Q_3 = -\sin \theta_3 = 0. \) Thus,
\[ \Delta \eta = (-R_1^2 + X_1 Q_1) \cdot [\Delta c_0, \Delta c_1]^T = X_1 Q_1 \cdot [\Delta c_0, \Delta c_1]^T = -1.0 \Delta c_0, \]
\[ \Delta \theta = -\frac{1}{\eta_1} (X_1 P_1 + R_1 Q_1) \cdot [\Delta c_0, \Delta c_1]^T = -\frac{1}{\eta_1} X_1 P_1 \cdot [\Delta c_0, \Delta c_1]^T = 1.1547 \Delta c_0 + 1.1547 \Delta c_1, \]
\[ \Delta \eta_3 = (-R_3^2 + X_3 Q_3) \cdot \Delta d_0 = -\Delta d_0, \Delta \theta_3 = -\frac{1}{R_3} (X_3 P_3 + R_3 Q_3) \cdot \Delta d_0 = -\frac{1}{R_3} R_3 Q_3 \cdot \Delta d_0 = 0. \]

4. 5 points. Mitra 12.10(a)
Solution to part (a): We are given
\[ H(z) = \frac{(z + 4)(z - 1)}{(z + 0.4)(z + 0.2)} \]
and we do some long division to write
\[ H(z) = 1 + \frac{2.4z - 4.08}{z^2 + 0.6z + 0.08} = A + \frac{Cz + D}{z^2 + bz + d} = H_1(z) + H_2(z). \]

We can use the algebraic techniques in Section 12.5.5 to compute the normalized output noise variance as expressed in equation (12.85). Note \( R = 2, \) hence the double sum in (12.85) will have four terms in it. Referring to table 12.4, we see that the contour integrals related to \( H_1(z)H_2(z)^{-1}z^{-1} \) and \( H_2(z)H_1(z)^{-1}z^{-1} \) will be zero. Hence, equation (12.85) will result in only two non-zero terms, one of the form of \( I_1 \) and the other of the form of \( I_3. \)

We can easily compute \( I_1 = 1. \) Plugging in our values for \( C, D, b, \) and \( d \) from above, we can also compute \( I_3 = 48.4565. \) Hence, the total normalized output noise variance is \( \sigma_{v,n}^2 = 49.4565. \) We can also confirm this via simulation, adapting the code provided in lecture.

\begin{verbatim}
% output noise variance via simulation
% DRB ECE503 Spring 2012
delta = sqrt(12); % quantizer step size so that input noise variance is one
num = poly([-4 1]);
den = poly([-0.4 -0.2]);
N = 1e5;

% generate input quantization noise sequence
e = rand(1,N)*delta-delta/2;
disp(['Input noise variance : ' num2str(var(e))]);

% filter
v = filter(num,den,e);
\end{verbatim}
When I run this code, I get

<table>
<thead>
<tr>
<th>Description</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Input noise variance</td>
<td>0.99845</td>
</tr>
<tr>
<td>Output noise variance</td>
<td>49.3297</td>
</tr>
<tr>
<td>Ratio</td>
<td>49.4063</td>
</tr>
</tbody>
</table>

which is very close to the analytical prediction.
5. 5 points. Mitra 12.13(a)

Solution to part (a):

**Cascade Structure #1**: \( G(z) = \frac{(1-0.6z^{-1})(1+0.3z^{-1})}{(1-0.2z^{-1})(1+0.9z^{-1})} \). The noise model of this structure is as below

\[
\begin{align*}
\text{x[n]} & \quad 0.2 \quad \text{z}^{-1} \quad -0.6 \quad e_1[n] \\
\text{e}_1[n] & \quad 0.3 \quad \text{z}^{-1} \quad e_4[n] \\
\text{e}_2[n] & \quad 0.9 \quad e_3[n] \\
\text{y[n]} &
\end{align*}
\]

The noise transfer function from the noise source \( e_1[n] \) to the filter output is
\[
G_1(z) = \frac{(z-0.6)(z+0.3)}{(z-0.2)(z+0.9)} = 1 + \frac{-0.9091}{z-0.2} + \frac{0.9091}{z+0.9}. 
\]

The corresponding normalized noise variance at the output is
\[
\sigma^2_{1,n} = 1 + \frac{(-0.9091)^2}{1 - (-0.2)^2} + \frac{(0.9091)^2}{1 - (0.9)^2} + 2 \times \frac{(-0.9091) \times 0.9091}{1 - 0.9 \times (-0.2)} = 4.8098.
\]

Output of Program 12_4.m is 4.8098.

The noise transfer function from the noise sources \(e_2[n]\) and \(e_3[n]\) to the filter output is
\[
G_2(z) = \frac{z + 0.3}{z + 0.9} - 1 - \frac{0.6}{z + 0.9}.
\]

The normalized noise variance at the output due to each of these noise sources is
\[
\sigma^2_{2,n} = 1 + \frac{(0.6)^2}{1 - (0.9)^2} = 2.8947.
\]

Output of Program 12_4.m is 2.8947.

The noise transfer function from the noise source \(e_4[n]\) to the filter output is \(G_4(z) = 1\).

The corresponding normalized noise variance at the output is \(\sigma^2_{4,n} = 1\).

Hence the total normalized noise variance at the output is \(\sigma^2_n = \sigma^2_{1,n} + 2\sigma^2_{2,n} + \sigma^2_{4,n} = 11.5092\).

**Cascade Structure #2:** \(G(z) = \frac{(1 + 0.3z^{-1})(1 - 0.6z^{-1})}{(1 - 0.2z^{-1})(1 + 0.9z^{-1})}\). The noise model of this structure is as below:

![Diagram](image)

\[
\sigma^2_{1,n} = 4.8098 \text{ as in Structure#1.}
\]

The noise transfer function from the noise sources \(e_2[n]\) and \(e_3[n]\) to the filter output is
\[
G_2(z) = \frac{z - 0.6}{z + 0.9} = 1 + \frac{-1.5}{z + 0.9}. \text{ Hence, } \sigma^2_{2,n} = 1 + \frac{(-1.5)^2}{1 - (0.9)^2} = 12.8421.
\]

Output of Program 12_4.m is 12.8421.

The noise transfer function from the noise source \(e_4[n]\) to the filter output is \(G_4(z) = 1\).

The corresponding normalized noise variance at the output is \(\sigma^2_{4,n} = 1\).

Hence the total normalized noise variance at the output is \(\sigma^2_n = \sigma^2_{1,n} + 2\sigma^2_{2,n} + \sigma^2_{4,n} = 31.4944\).

**Cascade Structure #3:** \(G(z) = \frac{(1 + 0.3z^{-1})(1 - 0.6z^{-1})}{(1 + 0.9z^{-1})(1 - 0.2z^{-1})}\). The noise model of this structure is as below:
\( \sigma_{1,n}^2 = 4.8098 \) as in Structure #1.

The noise transfer function from the noise sources \( e_2[n] \) and \( e_3[n] \) to the filter output is

\[
G_2(z) = \frac{z - 0.6}{z - 0.2} = 1 + \frac{-0.4}{z - 0.2}. \quad \text{Hence, } \sigma_{2,n}^2 = 1 + \frac{(-0.4)^2}{1 - (-0.2)^2} = 1.1667.
\]

Output of Program 12_4.m is 1.1667.

The noise transfer function from the noise source \( e_4[n] \) to the filter output is \( G_4(z) = 1 \).

The corresponding normalized noise variance at the output is \( \sigma_{4,n}^2 = 1 \).

Hence the total normalized noise variance at the output is \( \sigma_n^2 = \sigma_{1,n}^2 + 2\sigma_{2,n}^2 + \sigma_{4,n}^2 = 8.1432 \).

**Cascade Structure #4:** \( G(z) = \frac{(1 - 0.6z^{-1})(1 + 0.3z^{-1})}{(1 + 0.9z^{-1})(1 - 0.2z^{-1})} \). The noise model of this structure is as below:

\( \sigma_{1,n}^2 = 4.8098 \) as in Structure #1.

The noise transfer function from the noise sources \( e_2[n] \) and \( e_3[n] \) to the filter output is

\[
G_2(z) = \frac{z + 0.3}{z - 0.2} = 1 + \frac{0.5}{z - 0.2}. \quad \text{Hence, } \sigma_{2,n}^2 = 1 + \frac{(0.5)^2}{1 - (-0.2)^2} = 1.2604.
\]

Output of Program 12_4.m is 1.2604.

The noise transfer function from the noise source \( e_4[n] \) to the filter output is \( G_4(z) = 1 \).

The corresponding normalized noise variance at the output is \( \sigma_{4,n}^2 = 1 \).

Hence the total normalized noise variance at the output is \( \sigma_n^2 = \sigma_{1,n}^2 + 2\sigma_{2,n}^2 + \sigma_{4,n}^2 = 8.3306 \).