ECE503 Homework Assignment Number 10 Solution

1. 5 points total.

   (a) 3 points. Analytically confirm the result in equation (12.47), i.e. calculate the matrix/vector products and apply any useful trigonometric identities to arrive at the final result in Example 12.2.
   Solution: We just do a little linear algebra and apply the necessary trig identities as follows:

$$\begin{aligned}
\begin{bmatrix}\Delta r\\ \Delta\theta\end{bmatrix} &= \begin{bmatrix}\frac{1}{2r} & 0\\ \frac{1}{2r^2\tan\theta} & -\frac{1}{2r\sin\theta}\end{bmatrix}\begin{bmatrix}2r\cos\theta & 2r\sin\theta\\ 2 & 0\end{bmatrix}\begin{bmatrix}\Delta\alpha\\ \Delta\beta\end{bmatrix}\\
&= \begin{bmatrix}\cos\theta & \sin\theta\\ \frac{1}{r}\frac{\cos\theta}{\tan\theta}-\frac{1}{r\sin\theta} & \frac{1}{r}\frac{\sin\theta}{\tan\theta}\end{bmatrix}\begin{bmatrix}\Delta\alpha\\ \Delta\beta\end{bmatrix}\\
&= \begin{bmatrix}\cos\theta & \sin\theta\\ \frac{1}{r}\frac{\cos^2\theta}{\sin\theta}-\frac{1}{r\sin\theta} & \frac{1}{r}\cos\theta\end{bmatrix}\begin{bmatrix}\Delta\alpha\\ \Delta\beta\end{bmatrix}\\
&= \begin{bmatrix}\cos\theta & \sin\theta\\ \frac{1-\sin^2\theta}{r\sin\theta}-\frac{1}{r\sin\theta} & \frac{1}{r}\cos\theta\end{bmatrix}\begin{bmatrix}\Delta\alpha\\ \Delta\beta\end{bmatrix}\\
&= \begin{bmatrix}\cos\theta & \sin\theta\\ \frac{-\sin^2\theta}{r\sin\theta} & \frac{1}{r}\cos\theta\end{bmatrix}\begin{bmatrix}\Delta\alpha\\ \Delta\beta\end{bmatrix}\\
&= \begin{bmatrix}\cos\theta & \sin\theta\\ -\frac{1}{r}\sin\theta & \frac{1}{r}\cos\theta\end{bmatrix}\begin{bmatrix}\Delta\alpha\\ \Delta\beta\end{bmatrix}
\end{aligned}$$

   This is the desired result.

   (b) 2 points. Numerically verify the result for a particular choice of pole locations $r = 0.9$, $\theta = \pi/4$, and coefficient quantization errors $\Delta\alpha = 0.01$ and $\Delta\beta = -0.01$. In other words, numerically compute the *exact* radial and angular displacement of the poles from the transfer function with quantized and unquantized α and β and then use equation (12.47) to compute the analytically predicted approximate displacement of the poles. Is the analytical prediction accurate?
   Solution: Here is my code.

```
% ECE503 Spring 2012

% original unquantized poles, coupled form parameters (see p.674)
r = 0.9;
theta = pi/4;
alpha = r*cos(theta);
beta = r*sin(theta);
lam = roots([1 -2*alpha (alpha^2+beta^2)]); % unquantized roots

% quantized roots
```

```
delta_alpha = 0.01;
delta_beta = -0.01;
alphahat = alpha+delta_alpha;
betahat = beta+delta_beta;
lamhat = roots([1 -2*alphahat (alphahat^2+betahat^2)]); % exact quantized roots
rhat = abs(lamhat);
thetahat = angle(lamhat);
delta_r_exact = rhat - [r;r]
delta_theta_exact = thetahat - [theta;-theta]

% theoretical prediction via approximation
delta_r_approx = cos(theta)*delta_alpha + sin(theta)*delta_beta
delta_theta_approx = -(1/r)*sin(theta)*delta_alpha + (1/r)*cos(theta)*delta_beta
```

Here are my results.

```
delta_r_exact =

   1.0e-03 *

    0.1111
    0.1111


delta_theta_exact =

   -0.0157
    0.0157


delta_r_approx =

   1.7347e-18


delta_theta_approx =

   -0.0157
```

The approximation says the roots should not change radially (the exact solution shows that the radius of the roots changes very slightly, increasing by about 10^{-4}). The approximation also very accurately predicts the angular change of the roots caused by the slight changes in the coupled form parameters α and β. The approximation says that the angle of the first root will change by about -0.0157 (which it does). Since the roots have to appear as complex conjugates, the angle of the second root changes by $+0.0157$. Hence, the analytical prediction is quite accurate in this case.

2. 4 points. Mitra 12.2
Solution: Since we are concerned with pole sensitivity here, note that the denominator of both the highpass and lowpass filters in Figure 8.34 is just $B(z) = z - \alpha$. Since the coefficient α is real, there is only one pole at $z = \alpha = re^{j\theta}$ with $r = |\alpha|$ and $\theta = 0$ if $\alpha > 0$ or $\theta = \pi$ if $\alpha < 0$.

When α is quantized to $\hat{\alpha}$, we have $\hat{r} = |\hat{\alpha}|$ and $\hat{\theta} = 0$ if $\hat{\alpha} > 0$ or $\hat{\theta} = \pi$ if $\hat{\alpha} < 0$.

As long as the quantization of α is such that $\hat{\alpha}$ has the same sign as α (which will be the case for any reasonable quantization scheme), we can say that $\Delta\theta = 0$ and

$$\Delta r = \begin{cases} \Delta\alpha & \alpha > 0 \\ -\Delta\alpha & \alpha < 0. \end{cases}$$

3. 6 points. Mitra 12.6
Solution to part (a):

(a) For direct form implementation $B(z) = (z - z_1)(z - z_2)(z - z_3)$, where $z_1 = r_1 e^{j\theta_1}$, $z_2 = r_2 e^{j\theta_2}$, and $z_3 = r_3 e^{j\theta_3}$. Thus, $B(z) = (z^2 - 2r_1\cos\theta_1 z + r_1^2)(z - r_3)$ $= (z^2 - 0.5z + 0.25)(z + 0.75)$. This implies, $2r_1\cos\theta_1 = 0.5$, $r_1^2 = 0.25$, $r_3 = -0.75$, and $\theta_3 = \pi$. Thus, $r_1 = \sqrt{0.25} = 0.5$ and $\cos\theta_1 = \dfrac{0.5}{2 \times 0.5} = 0.5$. Now, $\dfrac{1}{B(z)} = \dfrac{1}{(z^2 - 0.5z + 0.25)(z + 0.75)}$
$= \dfrac{-0.4211 - j0.972}{z - 0.25 - j0.433} + \dfrac{-0.4211 + j0.972}{z - 0.25 + j0.433} + \dfrac{0.8421}{z + 0.75}$.

$\mathbf{P}_1 = \begin{bmatrix}\cos\theta_1 & r_1 & r_1^2\cos\theta_1\end{bmatrix} = \begin{bmatrix}0.5 & 0.5 & 0.125\end{bmatrix}$,

$\mathbf{Q}_1 = \begin{bmatrix}\sin\theta_1 & 0 & r_1^2\sin\theta_1\end{bmatrix} = \begin{bmatrix}0.866 & 0 & 0.2165\end{bmatrix}$, $R_1 = -0.4211$, and $X_1 = 0.972$. Likewise,

$\mathbf{P}_3 = \begin{bmatrix}\cos\theta_3 & r_3 & r_3^2\cos\theta_3\end{bmatrix} = \begin{bmatrix}-1 & 0.75 & 0.5625\end{bmatrix}$,

$\mathbf{Q}_3 = \begin{bmatrix}\sin\theta_3 & 0 & r_3^2\sin\theta_3\end{bmatrix} = \begin{bmatrix}0 & 0 & 0\end{bmatrix}$, $R_3 = 0.8421$, and $X_3 = 0$.

Thus, $\Delta r_1 = (-R_1\mathbf{P}_1 + X_1\mathbf{Q}_1)\cdot \Delta\mathbf{B} = 1.0523\Delta b_0 + 0.2105\Delta b_1 + 0.2631\Delta b_2$,

$\Delta\theta_1 = -\dfrac{1}{r_1}(X_1\mathbf{P}_1 + R_1\mathbf{Q}_1)\cdot \Delta\mathbf{B} = -0.5509\Delta b_0 - 0.5509\Delta b_1 - 0.1377\Delta b_2$,

$\Delta r_3 = (-R_3\mathbf{P}_3 + X_3\mathbf{Q}_3)\cdot \Delta\mathbf{B} = 0.8421\Delta b_0 - 0.6316\Delta b_1 + 0.4737\Delta b_2$, and

$\Delta\theta_3 = -\dfrac{1}{r_3}(X_3\mathbf{P}_3 + R_3\mathbf{Q}_3)\cdot \Delta\mathbf{B} = 0$.

Solution to part (b):

(b) Cascade form: $B(z) = (z^2 + c_1 z + c_0)(z + d_0) = B_1(z)B_2(z)$, where $B_1(z) = z^2 + c_1 z + c_0 = z^2 - 0.5z + 0.25 = (z - r_1 e^{j\theta_1})(z - r_1 e^{-j\theta_1}) = z^2 - 2r\cos\theta_1 z + r_1^2$ and $B_2(z) = z + d_0 = z = 0.75 = z - r_3 e^{j\theta_3}$. Comparing we get $2r_1\cos\theta_1 = 0.5$, $r_1^2 = 0.25$,

$r_3 = 0.75,\ \theta_3 = \pi$. Solving the first two equations we get $r_1 = \sqrt{0.25} = 0.5$ and $\cos\theta_1 = \dfrac{0.5}{2\sqrt{0.25}} = 0.5$.

Now, $\dfrac{1}{B(z_1)} = \dfrac{-j1.1547}{z-0.25-j0.433} + \dfrac{j1.1547}{z-0.25+j0.433}$. Hence, $R_1 = 0$ and $X_1 = -1.1547$.

$\mathbf{P}_1 = [\cos\theta_1 \quad r_1] = [0.5 \quad 0.5], \mathbf{Q}_1 = [-\sin\theta_1 \quad 0] = [0.866 \quad 0]$.

Next, $\dfrac{1}{B_2(z)} = \dfrac{1}{z+0.75}$. Hence, $R_3 = 1$ and $X_3 = 0$. Here, $\mathbf{P}_3 = \cos\theta_3 = -1$, and $\mathbf{Q}_3 = -\sin\theta_3 = 0$. Thus,

$\Delta r_1 = (-R_1\mathbf{P}_1 + X_1\mathbf{Q}_1)\cdot[\Delta c_0 \quad \Delta c_1]^t = X_1\mathbf{Q}_1\cdot[\Delta c_0 \quad \Delta c_1]^t = -1.0\Delta c_0,$

$\Delta\theta_1 = -\dfrac{1}{r_1}(X_1\mathbf{P}_1 + R_1\mathbf{Q}_1)\cdot[\Delta c_0 \quad \Delta c_1]^t = -\dfrac{1}{r_1}\cdot X_1\mathbf{P}_1\cdot[\Delta c_0 \quad \Delta c_1]^t = 1.1547\Delta c_0 + 1.1547\Delta c_1,$

$\Delta r_3 = (-R_3\mathbf{P}_3 + X_3\mathbf{Q}_3)\cdot\Delta d_0 = -\Delta d_0, \Delta\theta_3 = -\dfrac{1}{r_3}(X_3\mathbf{P}_3 + R_3\mathbf{Q}_3)\cdot\Delta d_0 = -\dfrac{1}{r_3}R_3\mathbf{Q}_3\cdot\Delta d_0 = 0.$

4. 5 points. Mitra 12.10(a)
Solution to part (a): We are given

$$H(z) = \frac{(z+4)(z-1)}{(z+0.4)(z+0.2)}$$

and we do some long division to write

$$H(z) = 1 + \frac{2.4z - 4.08}{z^2 + 0.6z + 0.08} = A + \frac{Cz + D}{z^2 + bz + d} = H_1(z) + H_2(z).$$

We can use the algebraic techniques in Section 12.5.5 to compute the normalized output noise variance as expressed in equation (12.85). Note $R = 2$, hence the double sum in (12.85) will have four terms in it. Referring to table 12.4, we see that the contour integrals related to $H_1(z)H_2(z^{-1})z^{-1}$ and $H_2(z)H_1(z^{-1})z^{-1}$ will be zero. Hence, equation (12.85) will result in only two non-zero terms, one of the form of I_1 and the other of the form I_3.

We can easily compute $I_1 = 1$. Plugging in our values for C, D, b, and d from above, we can also compute $I_3 = 48.4565$. Hence, the total normalized output noise variance is $\sigma^2_{v,n} = 49.4565$. We can also confirm this via simulation, adapting the code provided in lecture.

```
% output noise variance via simulation
% DRB ECE503 Spring 2012
delta = sqrt(12);       % quantizer step size so that input noise variance is one
num = poly([-4 1]);
den = poly([-0.4 -0.2]);
N = 1e5;

% generate input quantization noise sequence
e = rand(1,N)*delta-delta/2;
disp(['Input noise variance        : ' num2str(var(e))]);

% filter
v = filter(num,den,e);
```

```
% compute output noise variance
disp(['Output noise variance        : ' num2str(var(v))]);
disp(['Ratio                        : ' num2str(var(v)/var(e))]);
```

When I run this code, I get

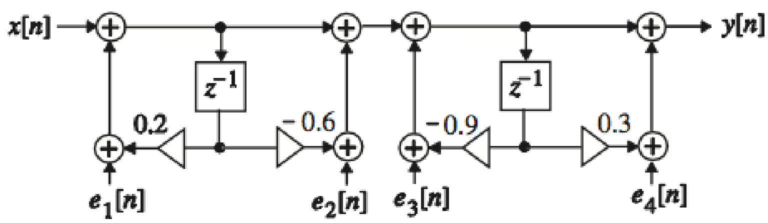
```
Input noise variance        : 0.99845
Output noise variance       : 49.3297
Ratio                       : 49.4063
```

which is very close to the analytical prediction.

5. 5 points. Mitra 12.13(a)

Solution to part (a):

Cascade Structure #1: $G(z) = \dfrac{(1-0.6z^{-1})(1+0.3z^{-1})}{(1-0.2z^{-1})(1+0.9z^{-1})}$. The noise model of this structure is as below



The noise transfer function from the noise source $e_1[n]$ to the filter output is

$$G_1(z) = \frac{(z-0.6)(z+0.3)}{(z-0.2)(z+0.9)} = 1 + \frac{-0.9091}{z-0.2} + \frac{0.9091}{z+0.9}.$$

The corresponding normalized noise variance at the output is

$$\sigma_{1,n}^2 = 1 + \frac{(-0.9091)^2}{1-(-0.2)^2} + \frac{(0.9091)^2}{1-(0.9)^2} + \frac{2\times(-0.9091)\times 0.9091}{1-0.9\times(-0.2)} = 4.8098.$$

Output of Program 12_4.m is 4.8098.

The noise transfer function from the noise sources $e_2[n]$ and $e_3[n]$ to the filter output is

$$G_2(z) = \frac{z+0.3}{z+0.9} = 1 - \frac{0.6}{z+0.9}.$$

The normalized noise variance at the output due to each of these noise sources is

$$\sigma_{2,n}^2 = 1 + \frac{(0.6)^2}{1-(0.9)^2} = 2.8947.$$

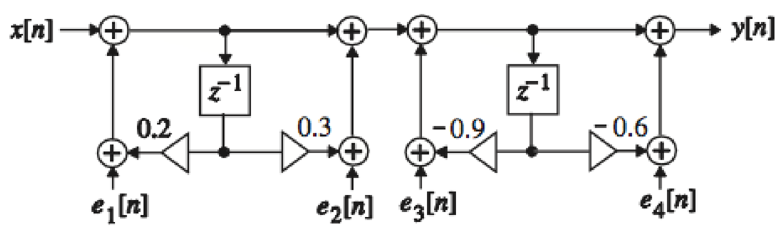
Output of Program 12_4.m is 2.8947.

The noise transfer function from the noise source $e_4[n]$ to the filter output is $G_4(z) = 1$.

The corresponding normalized noise variance at the output is $\sigma_{4,n}^2 = 1$.

Hence the total normalized noise variance at the output is $\sigma_n^2 = \sigma_{1,n}^2 + 2\sigma_{2,n}^2 + \sigma_{4,n}^2 = 11.5092$.

Cascade Structure #2: $G(z) = \dfrac{(1+0.3z^{-1})(1-0.6z^{-1})}{(1-0.2z^{-1})(1+0.9z^{-1})}$. The noise model of this structure is as below:



$\sigma_{1,n}^2 = 4.8098$ as in Structure#1.

The noise transfer function from the noise sources $e_2[n]$ and $e_3[n]$ to the filter output is

$G_2(z) = \dfrac{z-0.6}{z+0.9} = 1 + \dfrac{-1.5}{z+0.9}$. Hence, $\sigma_{2,n}^2 = 1 + \dfrac{(-1.5)^2}{1-(0.9)^2} = 12.8421$.

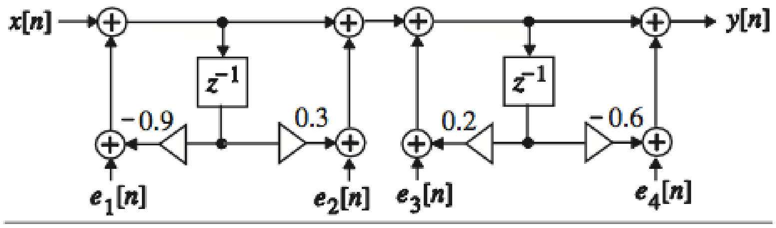
Output of Program 12_4.m is 12.8421.

The noise transfer function from the noise source $e_4[n]$ to the filter output is $G_4(z) = 1$.

The corresponding normalized noise variance at the output is $\sigma_{4,n}^2 = 1$.

Hence the total normalized noise variance at the output is $\sigma_n^2 = \sigma_{1,n}^2 + 2\sigma_{2,n}^2 + \sigma_{4,n}^2 = 31.494$.

Cascade Structure #3: $G(z) = \dfrac{(1+0.3z^{-1})(1-0.6z^{-1})}{(1+0.9z^{-1})(1-0.2z^{-1})}$. The noise model of this structure is as below:



$\sigma_{1,n}^2 = 4.8098$ as in Structure#1.

The noise transfer function from the noise sources $e_2[n]$ and $e_3[n]$ to the filter output is

$G_2(z) = \frac{z-0.6}{z-0.2} = 1 + \frac{-0.4}{z-0.2}$. Hence, $\sigma_{2,n}^2 = 1 + \frac{(-0.4)^2}{1-(-0.2)^2} = 1.1667.$

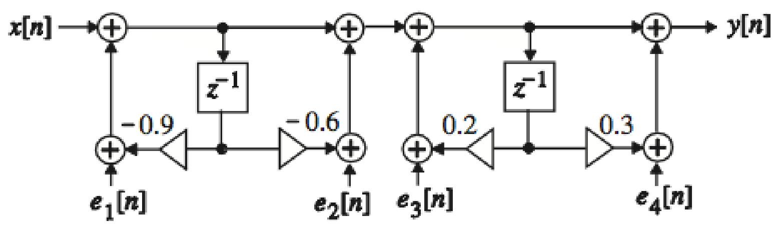
Output of Program 12_4.m is 1.1667.

The noise transfer function from the noise source $e_4[n]$ to the filter output is $G_4(z) = 1$.

The corresponding normalized noise variance at the output is $\sigma_{4,n}^2 = 1$.

Hence the total normalized noise variance at the output is $\sigma_n^2 = \sigma_{1,n}^2 + 2\sigma_{2,n}^2 + \sigma_{4,n}^2 = 8.1432.$

Cascade Structure #4: $G(z) = \frac{(1-0.6z^{-1})(1+0.3z^{-1})}{(1+0.9z^{-1})(1-0.2z^{-1})}$. The noise model of this structure is as below:



$\sigma_{1,n}^2 = 4.8098$ as in Structure#1.

The noise transfer function from the noise sources $e_2[n]$ and $e_3[n]$ to the filter output is

$G_2(z) = \frac{z+0.3}{z-0.2} = 1 + \frac{0.5}{z-0.2}$. Hence, $\sigma_{2,n}^2 = 1 + \frac{(0.5)^2}{1-(-0.2)^2} = 1.2604.$

Output of Program 12_4.m is 1.2604.

The noise transfer function from the noise source $e_4[n]$ to the filter output is $G_4(z) = 1$.

The corresponding normalized noise variance at the output is $\sigma_{4,n}^2 = 1$.

Hence the total normalized noise variance at the output is $\sigma_n^2 = \sigma_{1,n}^2 + 2\sigma_{2,n}^2 + \sigma_{4,n}^2 = 8.3306.$