ECE503 Homework Assignment Number 1

Due by 8:50pm on Monday 23-Jan-2012

IMPORTANT: Please place your ECE mailbox number on all homework assignments. Your ECE mailbox number can be found on the course web page.

Make sure your reasoning and work are clear to receive full credit for each problem. Points will be deducted for a disorderly presentation of your solution. Please also refer to the course academic honesty policies regarding collaboration on homework assignments.

This assignment should be largely review from material you learned in an undergraduate discrete-time signals and systems course like ECE2312. It is recommended that you skim over all of the problems in Chapters 2 and 3. Most of the problems in these chapters should be straightforward with the background of an undergraduate course in discrete-time signals and systems.

1. 4 points. Mitra 2.4.

2. 3 points. Mitra 2.30. You can/should use Matlab to confirm your answers are correct at least for some particular choices of α.

3. 4 points. Mitra 2.47. You can/should use Matlab here to confirm your answers are correct.

4. 4 points. Mitra M2.4(a). Please be sure to comment your Matlab function. To generate a figure that looks exactly like Figure 2.22 in Mitra, you can use the command `subplot(4,2,n)` which generates a 4×2 array of little plots and puts your next plot in the nth position.

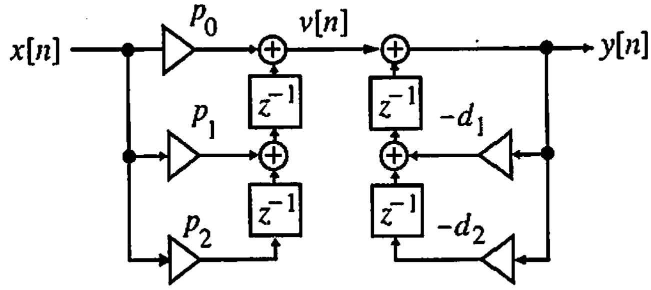
5. 3 points. Mitra 3.15.

6. 3 points. Mitra 3.61.

7. 4 points. Mitra 3.66.

1. Mitra 2.4

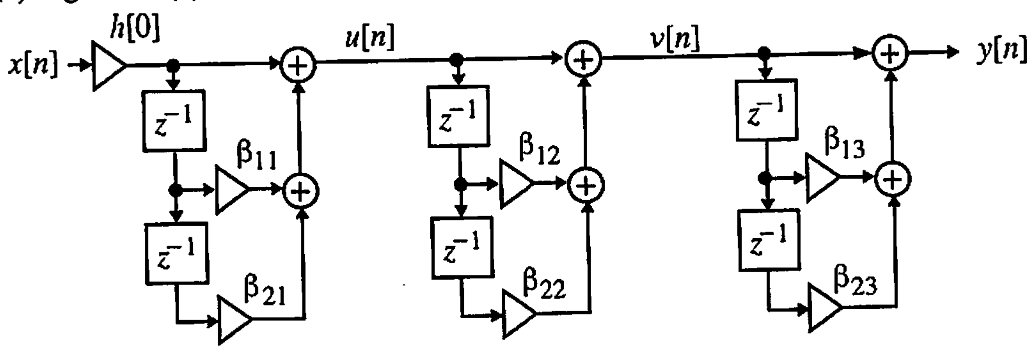
(a) The structure of Figure P2.1(a) is a cascade connection of two second-order structures. Reversing their order we arrive at the equivalent representation shown below:



Analyzing the first section we obtain $v[n] = p_0 x[n] + p_1 x[n-1] + p_0 x[n-2]$. Analyzing the second section we arrive at $y[n] = v[n] - d_1 y[n-1] - d_2 y[n-2]$, or equivalently $v[n] = y[n] + d_1 y[n-1] + d_2 y[n-2]$. Substituting the expression for v[n] derived from the analysis of the first section we arrive at the input-output relation of the structure of Figure 2.1(a): $y[n] + d_1 y[n-1] + d_2 y[n-2] = p_0 x[n] + p_1 x[n-1] + p_2 x[n-2]$.

(b) The structure of Figure 2.1(b) is precisely the figure shown in the solution of Part (a) given above. Hence, the input-output relation of the structure of Figure 2.1(b) is also: $y[n] + d_1 y[n-1] + d_2 y[n-2] = p_0 x[n] + p_1 x[n-1] + p_2 x[n-2]$.

(c) Figure 2.1(c) with internal variables labeled is shown below:



Analyzing the above figure we arrive at

$$u[n] = h[0](x[n] + \beta_{11} x[n-1] + \beta_{21} x[n-2]),$$
$$v[n] = u[n] + \beta_{12} u[n-1] + \beta_{22} u[n-2],$$
$$y[n] = v[n] + \beta_{13} v[n-1] + \beta_{23} v[n-2].$$

Substituting the expression for $v[n]$ in the last equation we arrive at

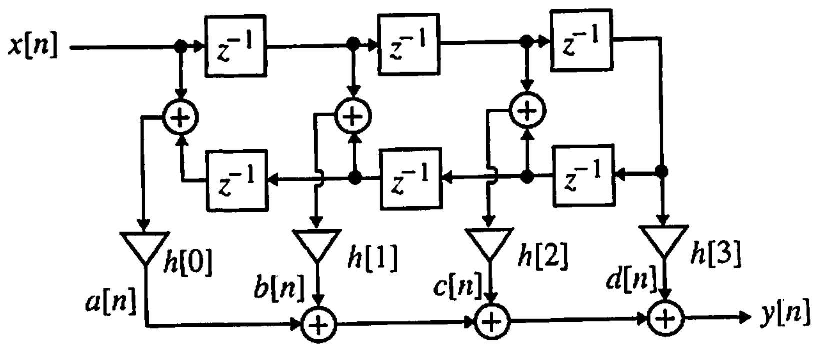
$$\begin{aligned} y[n] &= (u[n] + \beta_{12} u[n-1] + \beta_{22} u[n-2]) + \beta_{12}(u[n-1] + \beta_{12} u[n-2] + \beta_{22} u[n-3]) \\ &\quad + \beta_{23}(u[n-2] + \beta_{12} u[n-3] + \beta_{22} u[n-4]) \\ &= u[n] + (\beta_{12} + \beta_{13}) u[n-1] + (\beta_{22} + \beta_{12}\beta_{13} + \beta_{23}) u[n-2] \\ &\quad + (\beta_{13}\beta_{22} + \beta_{23}\beta_{12}) u[n-3] + \beta_{23}\beta_{22} u[n-4]. \end{aligned}$$

Finally, substituting the expression for u[n] from the first equation in the above equation and after some algebra we arrive at the input-output relation of the structure of Figure 2.1 (c).

(continued ...)

$$
\begin{aligned}
y[n] = {} & h[0]\big(x[n] + (\beta_{12} + \beta_{11} + \beta_{13})x[n-1]\big) \\
& + h[0]\big(\beta_{22} + \beta_{11}\beta_{12} + \beta_{21} + \beta_{12}\beta_{13} + \beta_{11}\beta_{13} + \beta_{23}\big)x[n-2] \\
& + h[0]\big(\beta_{11}\beta_{22} + \beta_{21}\beta_{12} + \beta_{22}\beta_{13} + \beta_{11}\beta_{12}\beta_{13} + \beta_{21}\beta_{13} + \beta_{12}\beta_{23} + \beta_{11}\beta_{23}\big)x[n-3] \\
& + h[0]\big(\beta_{21}\beta_{22} + \beta_{11}\beta_{22}\beta_{13} + \beta_{11}\beta_{22}\beta_{13} + \beta_{21}\beta_{12}\beta_{13} + \beta_{22}\beta_{23} + \beta_{11}\beta_{12}\beta_{23} + \beta_{21}\beta_{23}\big)x[n-4] \\
& + h[0]\big(\beta_{21}\beta_{22}\beta_{13} + \beta_{11}\beta_{22}\beta_{23} + \beta_{21}\beta_{12}\beta_{23}\big)x[n-5] \\
& + h[0]\big(\beta_{21}\beta_{22}\beta_{23}\big)x[n-6].
\end{aligned}
$$

(d) Figure 2.1(d) with internal variables labeled is shown below:



Analyzing the above figure we arrive at

$a[n] = h[0](x[n] + x[n-6]),$
$b[n] = h[1](x[n-1] + x[n-5]),$
$c[n] = h[2](x[n-2] + x[n-4]),$
$d[n] = h[3]x[n-3].$

We thus have

$$
\begin{aligned}
y[n] &= a[n] + b[n] + c[n] + d[n] \\
&= h[0](x[n] + x[n-6]) + h[1](x[n-1] + x[n-5]) \\
&= h[2](x[n-2] + x[n-4]) + h[3]x[n-3].
\end{aligned}
$$

2. Mitra 2.30

(a) $x_1[n] = \alpha^n \mu[n-1]$. Now, $\sum_{n=-\infty}^{\infty} |x_2[n]| = \sum_{n=1}^{\infty} |\alpha^n| = \sum_{n=1}^{\infty} |\alpha|^n = \frac{|\alpha|}{1-|\alpha|} < \infty$, since $|\alpha| < 1$. Hence, $x_1[n]$ is absolutely summable.

(b) $x_2[n] = \alpha^n \mu[n-1]$. Here, $\sum_{n=-\infty}^{\infty} |x_2[n]| = \sum_{n=1}^{\infty} |n\alpha^n| = \sum_{n=1}^{\infty} n|\alpha|^n = \frac{|\alpha|}{(1-|\alpha|)^2} < \infty$, since $|\alpha|^2 < 1$. Hence, $x_2[n]$ is absolutely summable.

(c) $x_3[n] = n^2 \alpha^n \mu[n-1]$. In this case, $\sum_{n=-\infty}^{\infty} |x_3[n]| = \sum_{n=1}^{\infty} |n^2 \alpha^n| = \sum_{n=1}^{\infty} n^2 |\alpha|^n$

$$= |\alpha| + 2^2|\alpha|^2 + 3^2|\alpha|^3 + 4^2|\alpha|^4 + \ldots$$

$$= (|\alpha| + |\alpha|^2 + |\alpha|^3 + |\alpha|^4 + \ldots) + 3(|\alpha|^2 + |\alpha|^3 + |\alpha|^4 + \ldots)$$

$$+ 5(|\alpha|^3 + |\alpha|^4 + |\alpha|^5 + \ldots) + 7(|\alpha|^4 + |\alpha|^5 + |\alpha|^6 + \ldots) + \ldots$$

$$= \frac{|\alpha|}{1-|\alpha|} + \frac{3|\alpha|^2}{1-|\alpha|} + \frac{5|\alpha|^3}{1-|\alpha|} + \frac{7|\alpha|^4}{1-|\alpha|} + \ldots$$

$$= \frac{1}{1-|\alpha|}\left(\sum_{n=1}^{\infty}(2n-1)|\alpha|^n\right) = \frac{1}{1-|\alpha|}\left(2\sum_{n=1}^{\infty} n|\alpha|^n - \sum_{n=1}^{\infty}|\alpha|^n\right)$$

$$= \frac{1}{1-|\alpha|}\left(\frac{2|\alpha|}{(1-|\alpha|)^2} - \frac{|\alpha|}{1-|\alpha|}\right) = \frac{|\alpha|(1+|\alpha|)}{(1-|\alpha|)^3} < \infty.$$

Hence, $x_3[n]$ is absolutely summable.

For $\alpha = 0.9$, we get

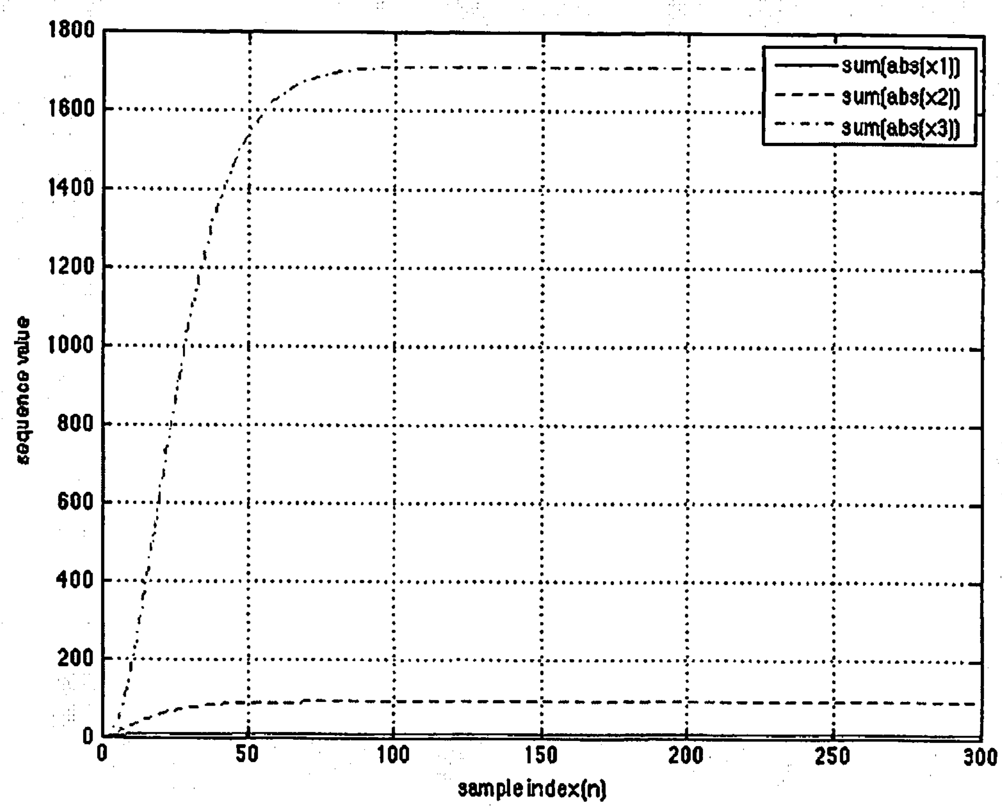
a) $\sum_{n=-\infty}^{\infty} |x_1[n]| = \frac{0.9}{1-0.9} = 9$

b) $\sum_{n=-\infty}^{\infty} |x_2[n]| = \frac{0.9}{(1-0.9)^2} = 90$

c) $\sum_{n=-\infty}^{\infty} |x_3[n]| = \frac{0.9\,(1+0.9)}{(1-0.9)^3} = \frac{1.71}{1\times 10^{-3}} = 1710$

see Matlab verification on next page

```
% Mitra problem 2.30
% numerical verification for particular values of alpha
% DRB 16-Jan-2012
% -------------------------------------------
% user parameters below
alpha = 0.9;
N = 300;            % length of sequence to generate (N+1 elements)
% -------------------------------------------

n = 0:N;
x1 = (alpha.^n).*(n>=1);
x2 = n.*(alpha.^n).*(n>=1);
x3 = (n.^2).*(alpha.^n).*(n>=1);

z1 = cumsum(abs(x1));       % compute the cumulative sum of the absolute values
z2 = cumsum(abs(x2));       % compute the cumulative sum of the absolute values
z3 = cumsum(abs(x3));       % compute the cumulative sum of the absolute values

plot(n,z1,n,z2,'--',n,z3,'-.');
xlabel('sample index (n)');
ylabel('sequence value');
grid on
legend('sum(abs(x1))','sum(abs(x2))','sum(abs(x3))');
```



>> z1(end) = 9.0000
>> z2(end) = 90.0000
>> z3(end) = 1.7100e+03

3. Mitra 2.47

In this problem we make use of the identity $\delta[n-m] \circledast \delta[n-r] = \delta[n-m-r]$.

(a) $y_1[n] = x_1[n] \circledast h_1[n]$

$$= (2\delta[n-1] - 2\delta[n+1]) \circledast (-\delta[n-2] - 1.5\delta[n] + \delta[n+3])$$

$$= -2\delta[n-1] \circledast \delta[n-2] - 3\delta[n-1] \circledast \delta[n] + 2\delta[n-1] \circledast \delta[n+3]$$

$$+ 2\delta[n+1] \circledast \delta[n-2] + 3\delta[n+1] \circledast \delta[n] - 2\delta[n+1] \circledast \delta[n+3]$$

$$= -2\delta[n-3] - 3\delta[n-1] + 2\delta[n+2] + 2\delta[n-1] + 3\delta[n+1] - 2\delta[n+4]$$

$$= -2\delta[n-3] - \delta[n-1] + 3\delta[n+1] + 2\delta[n+2] - 2\delta[n+4].$$

(b) $y_2[n] = x_2[n] \circledast h_2[n]$

$$= (3\delta[n-2] - \delta[n]) \circledast (3\delta[n-3] + 2\delta[n-1] - \delta[n+1])$$

$$= 9\delta[n-2] \circledast \delta[n-3] + 6\delta[n-2] \circledast \delta[n-1] - 3\delta[n-2] \circledast \delta[n+1]$$

$$- 3\delta[n] \circledast \delta[n-3] - 2\delta[n] \circledast \delta[n-1] + \delta[n] \circledast \delta[n+1]$$

$$= 9\delta[n-5] + 6\delta[n-3] - 3\delta[n-1] - 3\delta[n-3] - 2\delta[n-1] + \delta[n+1]$$

$$= 9\delta[n-5] + 3\delta[n-3] - 5\delta[n-1] + \delta[n+1].$$

(c) $y_3[n] = x_1[n] \circledast h_2[n]$

$$= (2\delta[n-1] - 2\delta[n+1]) \circledast (3\delta[n-3] + 2\delta[n-1] - \delta[n+1])$$

$$= 6\delta[n-1] \circledast \delta[n-3] + 4\delta[n-1] \circledast \delta[n-1] - 2\delta[n-1] \circledast \delta[n+1]$$

$$- 6\delta[n+1] \circledast \delta[n-3] - 4\delta[n+1] \circledast \delta[n-1] + 2\delta[n+1] \circledast \delta[n+1]$$

$$= 6\delta[n-4] + 4\delta[n-2] - 2\delta[n] - 6\delta[n-2] - 4\delta[n] + 2\delta[n+2]$$

$$= 6\delta[n-4] - 2\delta[n-2] - 6\delta[n] + 2\delta[n+2].$$

(d) $y_4[n] = x_2[n] \circledast h_1[n]$

$$= (3\delta[n-2] - \delta[n]) \circledast (-\delta[n-2] - 1.5\delta[n] + \delta[n+3])$$

$$= -3\delta[n-2] \circledast \delta[n-2] - 4.5\delta[n-2] \circledast \delta[n] + 3\delta[n-2] \circledast \delta[n+3]$$

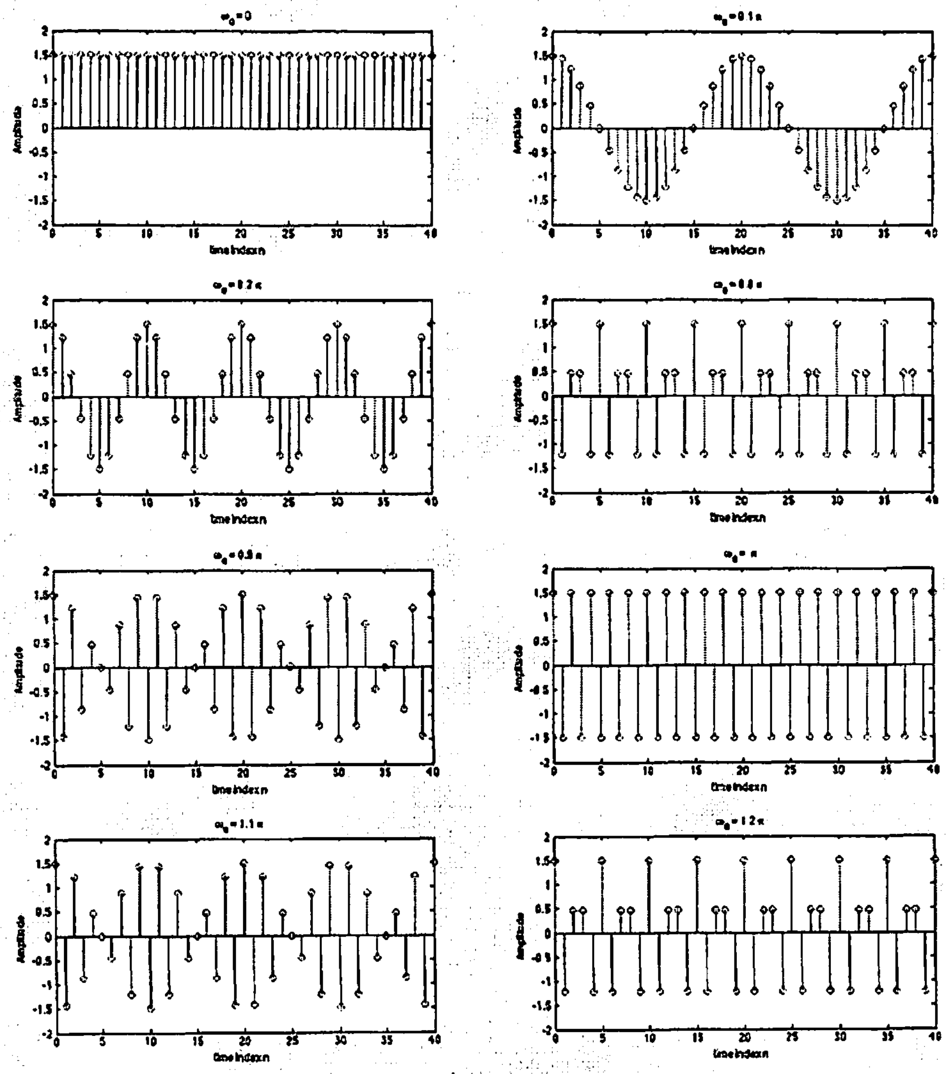
$$+ \delta[n] \circledast \delta[n-2] + 1.5\delta[n] \circledast \delta[n] - \delta[n] \circledast \delta[n+3]$$

$$= -3\delta[n-4] - 4.5\delta[n-2] + 3\delta[n+1] + \delta[n-2] + 1.5\delta[n] - \delta[n+3]$$

$$= -3\delta[n-4] - 3.5\delta[n-2] + 1.5\delta[n] + 3\delta[n+1] - \delta[n+3].$$

```
>> % Mitra problem 2.30
% DRB 16-Jan-2012

x1 = [-2 0 2]; % note this is delayed one sample from the x1 specified
x2 = [-1 0 3];
h1 = [1 0 0 -1.5 0 -1]; % note this is delayed three samples
h2 = [-1 0 2 0 3]; % note this delayed one sample

y1 = conv(x1,h1)
y2 = conv(x2,h2)
y3 = conv(x1,h2)
y4 = conv(x2,h1)

y1 =
                                                        (shift 1+3 samples)
  { -2     0     2     3     0    -1     0    -2 }

y2 =
                                                        (shift 0+1 samples)
  {  1     0    -5     0     3     0     9 }

y3 =
                                                        (shift 1+1 samples)
  {  2     0    -6     0    -2     0     6 }

y4 =
                                                        (shift 0+3 samples)
  { -1.0000         0    3.0000    1.5000         0   -3.5000         0   -3.0000 }
>>
```

```
% Mitra Problem M2.4(a)
% DRB 16-Jan-2012
% ----------------------------------
% user parameters
% ----------------------------------
L = 40;          % length
w0 = [0 0.1*pi 0.2*pi 0.8*pi 0.9*pi pi 1.1*pi 1.2*pi]; % normalized freq
A = 1.5;
phi = pi/2;
% ----------------------------------

mytitles = cell(length(w0));
mytitles{1} = '\omega_0 = 0';
mytitles{2} = '\omega_0 = 0.1\pi';
mytitles{3} = '\omega_0 = 0.2\pi';
mytitles{4} = '\omega_0 = 0.8\pi';
mytitles{5} = '\omega_0 = 0.9\pi';
mytitles{6} = '\omega_0 = \pi';
mytitles{7} = '\omega_0 = 1.1\pi';
mytitles{8} = '\omega_0 = 1.2\pi';

n = 0:L;         % sample indices
i1 = 0;
for w = w0,
    i1 = i1+1;
    subplot(4,2,i1);
    stem(n,A*sin(w*n+phi));
    axis([0 L -2 2]);
    xlabel('time index n');
    ylabel('Amplitude');
    title(mytitles{i1})
end
```


ω0 = 0
ω0 = π
Amplitude
time index n

5. Mitra 3.15

$$x[n] = A\alpha^n \cos(\omega_0 n + \phi)\mu[n] = A\alpha^n \left(\frac{e^{j\omega_0 n} e^{j\phi} + e^{-j\omega_0 n} e^{-j\phi}}{2} \right) \mu[n]$$

$= \frac{A}{2} e^{j\phi} \left(\alpha e^{j\omega_0} \right)^n \mu[n] + \frac{A}{2} e^{-j\phi} \left(\alpha e^{-j\omega_0} \right)^n \mu[n]$. Therefore, the DTFT of $x[n]$ is given by $X(e^{j\omega}) = \frac{A}{2} e^{j\phi} \frac{1}{1 - \alpha e^{j\omega_0} e^{-j\omega}} + \frac{A}{2} e^{-j\phi} \frac{1}{1 - \alpha e^{-j\omega_0} e^{-j\omega}}$.

6. Mitra 3.61

Since the continuous-time signal $x_a(t)$ is being sampled at 3.0 kHz rate, the sampled version of its i-th sinusoidal component with a frequency F_i will generate discrete-time sinusoidal signals with frequencies $F_i \pm 3000n, -\infty < n < \infty$. Hence, the frequencies F_{im} generated in the sampled version associated with the sinusoidal components present in are as follows:

$$F_1 = 300 \text{ Hz} \Rightarrow F_{1m} = 300, 2700, 3300, \ldots \text{ Hz}$$
$$F_2 = 500 \text{ Hz} \Rightarrow F_{2m} = 500, 2500, 3500, \ldots \text{ Hz}$$
$$F_3 = 1200 \text{ Hz} \Rightarrow F_{3m} = 1200, 1800, 4200, \ldots \text{ Hz}$$
$$F_4 = 2150 \text{ Hz} \Rightarrow F_{4m} = 850, 2150, 5150, \ldots \text{ Hz}$$
$$F_5 = 3500 \text{ Hz} \Rightarrow F_{5m} = 500, 3500, 6500, \ldots \text{ Hz}$$

After filtering by a lowpass filter with a cutoff at 900 Hz, the frequencies of the sinusoidal components in $y_a(t)$ are $300, 500, 850$ Hz.

7. Mitra 3.66 – for each part, we'll use the procedure covered in Lecture 1.

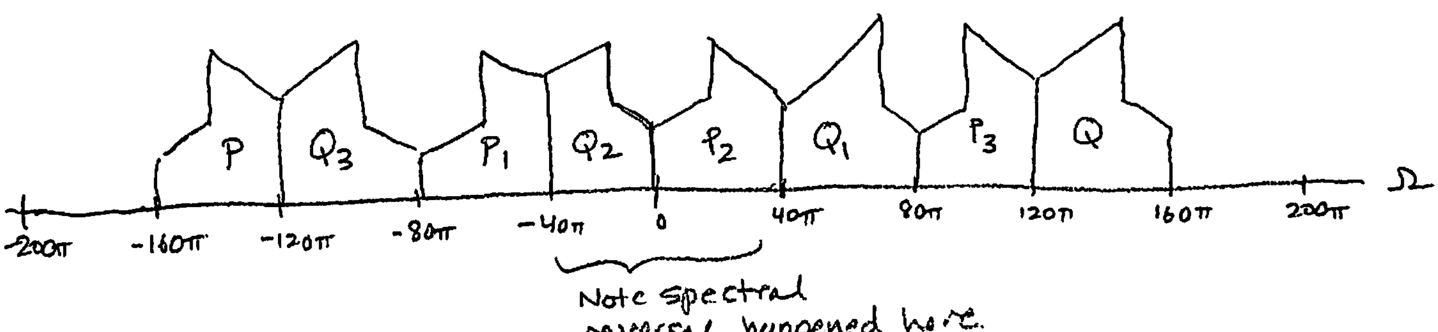
(a) $\Omega_1 = 120\pi$, $\Omega_2 = 160\pi$

$\Rightarrow B = 40\pi$

$\Omega_c = 140\pi$

replicas (m)	$\frac{2\Omega_c + B}{m+1}$	$\frac{2\Omega_c - B}{m}$	
1	160π	240π	
2	$106\frac{2}{3}\pi$	120π	
3	80π	80π	
~~4~~	~~64π~~	~~60π~~	won't work

$\Omega_T = 80\pi$ looks good. Let's sketch the spectrum at this sampling frequency (Note: $\Omega_T = 80\pi \geq 2B$, which is important).



(b) $\Omega_1 = 141\pi$, $\Omega_2 = 183\pi$

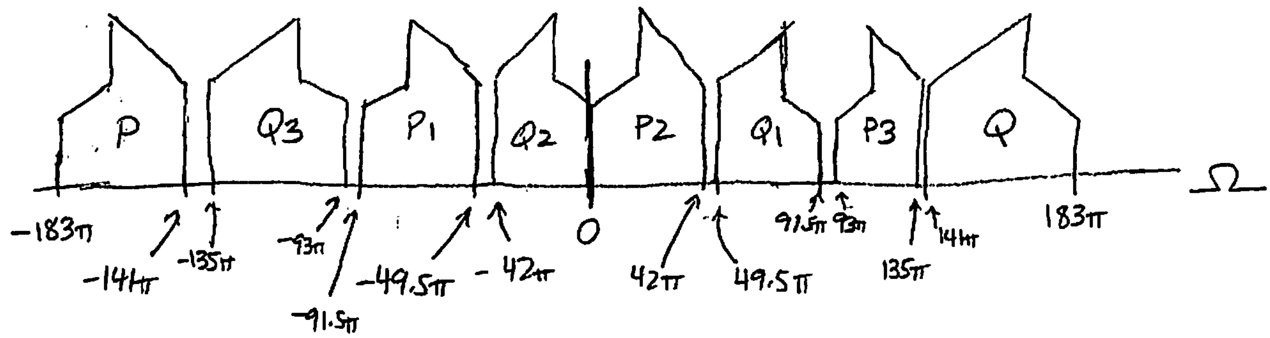
$B = 42\pi$

$\Omega_c = 162\pi$

replicas (m)	$\frac{2\Omega_c + B}{m+1}$	$\frac{2\Omega_c - B}{m}$	
1	183π	282π	
2	122π	141π	
3	91.5π	94π	
~~4~~	~~73.2π~~	~~70.5π~~	won't work.

$\Omega_T = 91.5\pi$ looks good and is $\geq 2B$.

sketch spectrum ... (continued)



spectral reversal present again

(c) $\Omega_1 = 168\pi$, $\Omega_2 = 220\pi$

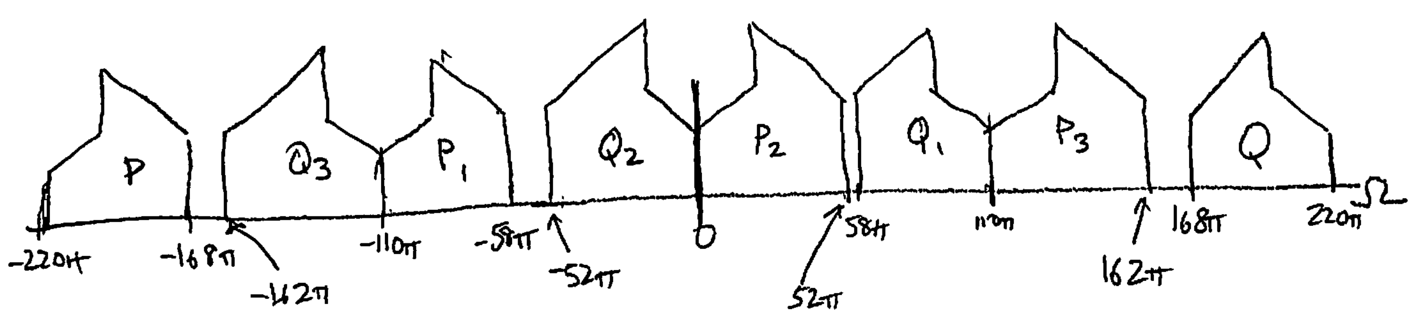
$B = 52\pi$

$\Omega_c = 194\pi$

replicas (m)	$\frac{2\Omega_c + B}{m+1}$	$\frac{2\Omega_c - B}{m}$	
1	220π	336π	
2	$146\frac{2}{3}\pi$	168π	
3	110π	112π	
~~4~~	~~88π~~	~~84π~~	won't work.

$\Omega_T = 110\pi \geq 2B$ looks good.

Sketch spectrum...



spectral reversal present again

end