ECE503 Homework Assignment Number 4 Solution

1. 3 points. Mitra 6.9.

   Solution:

   $v[n] = \alpha^{|n|} = \alpha^n \mu[n] + \alpha^{-n}\mu[-n-1]$. Now, $\mathcal{Z}\{\alpha^n \mu[n]\} = \dfrac{1}{1-\alpha z^{-1}}, |z| > |\alpha|$. (See Table 6.1) and $\mathcal{Z}\{\alpha^{-n}\mu[-n-1]\} = \sum_{n=-\infty}^{-1} \alpha^{-n} z^{-n} = \sum_{m=1}^{\infty} \alpha^m z^m = \sum_{m=0}^{\infty} \alpha^m z^m - 1 = \dfrac{1}{1-\alpha z} - 1$
   $= \dfrac{\alpha z}{1-\alpha z}, |\alpha z| < 1$. Therefore, $\mathcal{Z}\{v[n]\} = V(z) = \dfrac{1}{1-\alpha z^{-1}} + \dfrac{\alpha}{z^{-1}-\alpha} = \dfrac{(1-\alpha^2)z^{-1}}{(1-\alpha z^{-1})(z^{-1}-\alpha)}$ with its ROC given by $|\alpha| < |z| < |1/\alpha|$.

2. 4 points. Mitra 6.27.

   Solution to part (a):

   (a) $X_a(z) = \dfrac{7}{1+0.3z^{-1} - 0.1z^{-2}} = \dfrac{\rho_1}{1+0.5z^{-1}} + \dfrac{\rho_2}{1-0.2z^{-1}}$,

   where $\rho_1 = \left.\dfrac{7}{1-0.2z^{-1}}\right|_{z=-0.5} = 5$, $\rho_2 = \left.\dfrac{7}{1+0.5z^{-1}}\right|_{z=0.2} = 2$.

   Therefore, $X_a(z) = \dfrac{5}{1+0.5z^{-1}} + \dfrac{2}{1-0.2z^{-1}}$.

   There are three ROCs - $\mathcal{R}_1 : |z| < 0.2$, $\mathcal{R}_2 : 0.2 < |z| < 0.5$, $\mathcal{R}_3 : |z| > 0.5$.

   The inverse z–transform associated with the ROC $\mathcal{R}_1$ is a left-sided sequence:

   $\mathcal{Z}^{-1}\{X_a(z)\} = x_a[n] = \left(5(-0.5)^n + 2(0.2)^n\right)\mu[-n-1]$.

   The inverse z–transform associated with the ROC $\mathcal{R}_2$ is a two-sided sequence:

   $\mathcal{Z}^{-1}\{X_a(z)\} = x_a[n] = 5(-0.5)^n \mu[-n-1] + 2(0.2)^n \mu[n]$.

   The inverse z–transform associated with the ROC $\mathcal{R}_3$ is a right-sided sequence:

   $\mathcal{Z}^{-1}\{X_a(z)\} = x_a[n] = \left(5(-0.5)^n + 2(0.2)^n\right)\mu[n]$.

Solution to part (b):

(b) $X_b(z) = \dfrac{3z^2 + 1.8z + 1.28}{(z-0.5)(z+0.4)^2} = \dfrac{3z^{-1} + 1.8z^{-2} + 1.28z^{-3}}{(1-0.5z^{-1})(1+0.4z^{-1})^2}$

$$= K + \frac{\rho_1}{1-0.5z^{-1}} + \frac{\rho_2}{1+0.4z^{-1}} + \frac{\rho_3}{(1+0.4z^{-1})^2}.$$

$$K = X_b(0) = \frac{1.28}{-0.5 \times (0.4)^2} = -16,$$

$$\rho_1 = \left.\frac{3z^{-1} + 1.8z^{-2} + 1.28z^{-3}}{(1+0.4z^{-1})^2}\right|_{z=0.5} = 7.2346,$$

$$\rho_3 = \left.\frac{3z^{-1} + 1.8z^{-2} + 1.28z^{-3}}{(1-0.5z^{-1})}\right|_{z=-0.4} = -7.2222,$$

$$\rho_2 = \frac{1}{-0.4}\frac{d}{dz}\left.\left(\frac{3z^{-1} + 1.8z^{-2} + 1.28z^{-3}}{(1-0.5z^{-1})}\right)\right|_{z=-0.4} = 15.9877.$$ Hence,

$$X_b(z) = -16 + \frac{7.2346}{1-0.5z^{-1}} + \frac{-7.2222}{1+0.4z^{-1}} + \frac{15.9877}{(1+0.4z^{-1})^2}.$$

There are three ROCs - $\mathcal{R}_1 : |z| < 0.4$, $\mathcal{R}_2 : 0.4 < |z| < 0.5$, $\mathcal{R}_3 : |z| > 0.5$.

The inverse z–transform associated with the ROC $\mathcal{R}_1$ is a left-sided sequence:

$$Z^{-1}\{X_b(z)\} = x_b[n] = -16\delta[n] + 7.2346(0.5)^n \mu[-n-1] - 7.2222(-0.4)^n \mu[-n-1]$$
$$+15.9877(n+1)(-0.4)^n \mu[-n-1].$$

The inverse z–transform associated with the ROC $\mathcal{R}_2$ is a two-sided sequence:

$$Z^{-1}\{X_b(z)\} = x_b[n] = -16\delta[n] + 7.2346(0.5)^n \mu[n] - 7.2222(-0.4)^n \mu[-n-1]$$
$$+15.9877(n+1)(-0.4)^n \mu[-n-1].$$

The inverse z–transform associated with the ROC $\mathcal{R}_3$ is a right-sided sequence:

$$Z^{-1}\{X_b(z)\} = x_b[n] = -16\delta[n] + 7.2346(0.5)^n \mu[n] - 7.2222(-0.4)^n \mu[n]$$
$$+15.9877(n+1)(-0.4)^n \mu[n].$$

3. 4 points. Mitra 6.42.

Solution:

$$H(z) = H_1(z)H_3(z) + (1 + H_1(z))H_2(z)$$
$$= 11.06 + 8.51z^{-1} + 5.28z^{-2} + 5.12z^{-3} + 1.19z^{-4}.$$

4. 4 points. Mitra 6.44.

Solution to part (a):

(a) A partial-fraction expansion of $H(z)$ in z^{-1} using the M-file `residuez` yields $H(z) = -5 + \frac{4.0909}{1+0.4z^{-1}} + \frac{0.9091}{1-0.15z^{-1}}$. Hence, from Table 6.1 we have $h[n] = -5\delta[n] + 4.0909(-0.4)^n \mu[n] + 0.9091(0.15)^n \mu[n]$.

Solution to part (b):

(b) $x[n] = 2.1(0.4)^n \mu[n] + 0.3(-0.3)^n \mu[n]$. Its z–transform is thus given by $X(z) = \frac{2.1}{1-0.4z^{-1}} + \frac{0.3}{1+0.3z^{-1}} = \frac{2.4+0.51z^{-1}}{(1-0.4z^{-1})(1+0.3z^{-1})}, |z| > 0.4$. The z–transform of the output $y[n]$ is then given by

$$Y(z) = H(z)X(z) = \left[\frac{2.4+0.51z^{-1}}{(1-0.4z^{-1})(1+0.3z^{-1})}\right]\cdot\left[\frac{-1.5z^{-1}+0.3z^{-2}}{1+0.25z^{-1}-0.06z^{-2}}\right].$$

A partial-fraction expansion of $Y(z)$ in z^{-1} using the M-file `residuez` yields $Y(z) = \frac{9.2045}{1+0.4z^{-1}} - \frac{3.15}{1-0.4z^{-1}} - \frac{5}{1+0.3z^{-1}} - \frac{1.0545}{1-0.15z^{-1}}, |z| > 0.4$. Hence, from Table 6.1 we have $y[n] = \left(9.2045(-0.4)^n - 3.15(0.4)^n - 5(-0.3)^n - 1.0545(0.15)^n\right)\mu[n]$.

5. 3 points. Mitra 6.48 (a).

Solution to part (a):

Let the output of the predictor of Figure P6.4(a) be denoted by $E(z)$. Then analysis of this structure yields $E(z) = P(z)[U(z)+E(z)]$ and $U(z) = X(z) - E(z)$. From the first equation we have $E(z) = \frac{P(z)}{1-P(z)}U(z)$ which when substituted in the second equation yields $H(z) = \frac{U(z)}{X(z)} = 1 - P(z)$.

Analyzing Figure P6.3(b) we get $Y(z) = V(z) + P(z)Y(z)$ which leads to $G(z) = \frac{Y(z)}{V(z)} = \frac{1}{1-P(z)}$, which is seen to be the inverse of $H(z)$.

Hence, for $P(z) = h_1 z^{-1}$, we have $H(z) = 1 - h_1 z^{-1}$ and $G(z) = \frac{1}{1-h_1 z^{-1}}$.

6. 3 points. Mitra 6.73

Solution:

(a) $Y(z) = X(z) + \alpha X(z)z^{-M}$, therefore, $H(z) = \dfrac{Y(z)}{X(z)} = 1 + \alpha z^{-M}$ and $h[n] = \delta[n] + \alpha\delta[n-M]$.

(b) $G(z) = \dfrac{1}{H(z)} = \dfrac{1}{1+\alpha z^{-M}} = \sum_{k=0}^{\infty}(-1)^k \alpha^k z^{-kM}$ by long division. Therefore, $g[n] = \sum_{k=0}^{\infty}(-\alpha)^k \delta[n-kM]$.

(c) The ROC of the causal $g[n]$ is $|z| > \left|(-\alpha)^{1/M}\right|$. As long as $\left|(-\alpha)^{1/M}\right| < 1$, the ROC will contain the unit circle and the inverse system will be stable.

7. 4 points. Mitra 6.83

Solution: Since $x[n] = \alpha^n \mu[n]$, we know from a simple table lookup that $X(z) = \frac{1}{1-\alpha z^{-1}}$ with ROC $|z| > |\alpha|$. We now let

$$X[k] = X(z)|_{z=z_k=e^{j2\pi k/N}} = \frac{1}{1-\alpha e^{-j2\pi k/N}}$$

Even though the notation looks like the DFT here, this isn't the DFT of $\{x[n]\}$ because $\{x[n]\}$ is an infinite length sequence and the DFT is only used on finite-length sequences.

Now we save that result for a bit and form a periodic extension of $\{x[n]\}$ as follows

$$\begin{aligned}\tilde{x}[n] &= \sum_{\ell=-\infty}^{\infty} x[n+\ell N] \\ &= \sum_{\ell=-\infty}^{\infty} \alpha^{n+\ell N}\mu[n+\ell N]\end{aligned}$$

Let $n = mN + p$ with $m \in \mathbb{Z}$ and $p = 0, 1, \ldots, N-1$. Then we can write

$$\begin{aligned}\tilde{x}[mN+p] &= \sum_{\ell=-\infty}^{\infty} \alpha^{mN+p+\ell N}\mu[mN+p+\ell N] \\ &= \alpha^{mN+p}\sum_{\ell=-m}^{\infty} \alpha^{\ell N} \\ &= \alpha^{mN+p}\frac{\alpha^{-mN}}{1-\alpha^N} \\ &= \frac{\alpha^p}{1-\alpha^N}\end{aligned}$$

for $n = mN + p$ and $p = 0, 1, \ldots, N-1$. If you plot this, you will see a periodic sawtooth type of waveform. Now we take the DFT of one period of $\tilde{x}[n]$. We can set $m = 0$ so that

$\tilde{x}[n] = \tilde{x}[p]$ for $n = 0, \ldots, N-1$ and write

$$\begin{aligned}
\tilde{X}[k] &= \sum_{p=0}^{N-1} \tilde{x}[p] e^{-j2\pi kp/N} \\
&= \frac{1}{1-\alpha^N} \sum_{p=0}^{N-1} \alpha^p e^{-j2\pi kp/N} \\
&= \frac{1}{1-\alpha^N} \sum_{p=0}^{N-1} \beta^p \\
&= \frac{1}{1-\alpha^N} \sum_{p=0}^{N-1} \beta^p \\
&= \frac{1}{1-\alpha^N} \frac{1-\beta^N}{1-\beta} \\
&= \frac{1}{1-\alpha^N} \frac{1-\alpha^N e^{-j2\pi k}}{1-\alpha e^{-j2\pi k/N}} \\
&= \frac{1}{1-\alpha e^{-j2\pi k/N}} \\
&= X[k]
\end{aligned}$$

since $e^{-j2\pi k} = 1$ for all integer k. So this problem establishes an interesting relationship between the z-transform of an infinite length sequence sampled at specific values of z on the unit circle (corresponding to the DFT frequencies) and the actual DFT of one period of the periodically-extended sequence.