1. 3 points. Mitra 6.9.

Solution:

\[ v[n] = \alpha |n| = \alpha^n \mu(n) + \alpha^{-n} \mu(-n-1). \]

Now, \( Z\{\alpha^n \mu(n)\} = \frac{1}{1 - \alpha^{-1}} \), \( |z| > |\alpha| \). (See Table 6.1)

\[ Z\{\alpha^{-n} \mu(-n-1)\} = \sum_{n=-\infty}^{\infty} \alpha^{-n} z^{-n} = \sum_{m=1}^{\infty} \alpha^{-n} z^{-m} = \sum_{m=0}^{\infty} \alpha^{-m} z^{-m} - 1 = \frac{1}{1 - \alpha} - 1 \]

\[ = \frac{\alpha}{1 - \alpha}, \quad |\alpha| < 1. \]

Therefore, \( Z\{v[n]\} = V(z) = \frac{1}{1 - \alpha z^{-1}} + \frac{\alpha}{z^{-1} - \alpha} = \frac{(1 - \alpha^2)z^{-1}}{(1 - \alpha z^{-1})(z^{-1} - \alpha)} \)

with its ROC given by \( |\alpha| < |z| < 1/|\alpha| \).

2. 4 points. Mitra 6.27.

Solution to part (a):

\[ X_a(z) = \frac{7}{1 + 0.3z^{-1} - 0.1z^{-2}} = \frac{\rho_1}{1 + 0.5z^{-1}} + \frac{\rho_2}{1 - 0.2z^{-1}}, \]

where \( \rho_1 = \begin{vmatrix} 7 \\ 1 - 0.2z^{-1} \end{vmatrix}_{z=0.5} = 5, \quad \rho_2 = \begin{vmatrix} 7 \\ 1 + 0.5z^{-1} \end{vmatrix}_{z=0.2} = 2. \)

Therefore, \( X_a(z) = \frac{5}{1 + 0.5z^{-1}} + \frac{2}{1 - 0.2z^{-1}}. \)

There are three ROCs - \( R_1 : |z| < 0.2, \quad R_2 : 0.2 < |z| < 0.5, \quad R_3 : |z| > 0.5. \)

The inverse \( z \)-transform associated with the ROC \( R_1 \) is a left-sided sequence:

\( Z^{-1}\{X_a(z)\} = x_a[n] = (5(-0.5)^n + 2(0.2)^n)\mu[-n-1]. \)

The inverse \( z \)-transform associated with the ROC \( R_2 \) is a two-sided sequence:

\( Z^{-1}\{X_a(z)\} = x_a[n] = 5(-0.5)^n \mu[-n-1] + 2(0.2)^n \mu[n]. \)

The inverse \( z \)-transform associated with the ROC \( R_3 \) is a right-sided sequence:

\( Z^{-1}\{X_a(z)\} = x_a[n] = (5(-0.5)^n + 2(0.2)^n)\mu[n]. \)
Solution to part (b):

\[ X_b(z) = \frac{3z^2 + 1.8z + 1.28}{(z - 0.5)(z + 0.4)^2} = \frac{3z^{-1} + 1.8z^{-2} + 1.28z^{-3}}{(1 - 0.5z^{-1})(1 + 0.4z^{-1})^2} \]

\[ = K + \frac{\rho_1}{1 - 0.5z^{-1}} + \frac{\rho_2}{1 + 0.4z^{-1}} + \frac{\rho_3}{(1 + 0.4z^{-1})^2} \]

\[ K = X_b(0) = \frac{1.28}{-0.5 \times (0.4)^2} = -16, \]

\[ \rho_1 = \frac{3z^{-1} + 1.8z^{-2} + 1.28z^{-3}}{(1 + 0.4z^{-1})^2} \bigg|_{z=0.5} = 7.2346, \]

\[ \rho_3 = \frac{3z^{-1} + 1.8z^{-2} + 1.28z^{-3}}{(1 - 0.5z^{-1})} \bigg|_{z=-0.4} = -7.2222, \]

\[ \rho_2 = \frac{1}{-0.4} \left( \frac{3z^{-1} + 1.8z^{-2} + 1.28z^{-3}}{(1 - 0.5z^{-1})} \right) \bigg|_{z=-0.4} = -15.9877. \]

Hence,

\[ X_b(z) = -16 + \frac{7.2346}{1 - 0.5z^{-1}} + \frac{7.2222}{1 + 0.4z^{-1}} + \frac{15.9877}{(1 + 0.4z^{-1})^2}. \]

There are three ROCs - \( R_1 : |z| < 0.4, \) \( R_2 : 0.4 < |z| < 0.5, \) \( R_3 : |z| > 0.5. \)

The inverse \( z \)-transform associated with the ROC \( R_1 \) is a left-sided sequence:

\[ Z^{-1}\{X_b(z)\} = x_b[n] = -16\delta[n] + 7.2346(0.5)^n \mu[n-1] - 7.2222(-0.4)^n \mu[-n-1] + 15.9877(n+1)(-0.4)^n \mu[-n-1]. \]

The inverse \( z \)-transform associated with the ROC \( R_2 \) is a two-sided sequence:

\[ Z^{-1}\{X_b(z)\} = x_b[n] = -16\delta[n] + 7.2346(0.5)^n \mu[n] - 7.2222(-0.4)^n \mu[-n-1] + 15.9877(n+1)(-0.4)^n \mu[-n-1]. \]

The inverse \( z \)-transform associated with the ROC \( R_3 \) is a right-sided sequence:

\[ Z^{-1}\{X_b(z)\} = x_b[n] = -16\delta[n] + 7.2346(0.5)^n \mu[n] - 7.2222(-0.4)^n \mu[n] + 15.9877(n+1)(-0.4)^n \mu[n]. \]

3. 4 points. Mitra 6.42.

Solution:

\[ H(z) = H_1(z)H_3(z) + (1 + H_1(z))H_2(z) \]

\[ = 11.06 + 8.51z^{-1} + 5.28z^{-2} + 5.12z^{-3} + 1.19z^{-4}. \]
4. 4 points. Mitra 6.44.

Solution to part (a):

(a) A partial-fraction expansion of $H(z)$ in $z^{-1}$ using the M-file \texttt{residue} yields

$$H(z) = -5 + \frac{4.0909}{1+0.4z^{-1}} + \frac{0.9091}{1-0.15z^{-1}}.$$ 

Hence, from Table 6.1 we have

$$h[n] = -5\delta[n] + 4.0909(-0.4)^{n}\mu[n] + 0.9091(0.15)^{n}\mu[n].$$

Solution to part (b):

(b) $x[n] = 2.1(0.4)^{n}\mu[n] + 0.3(-0.3)^{n}\mu[n]$. Its $z$–transform is thus given by

$$X(z) = \frac{2.1}{1-0.4z^{-1}} + \frac{0.3}{1+0.3z^{-1}} = \frac{2.4+0.51z^{-1}}{(1-0.4z^{-1})(1+0.3z^{-1})}, |z|>0.4.$$ 

The $z$–transform of the output $y[n]$ is then given by

$$Y(z) = H(z)X(z) = \left[ \frac{2.4+0.51z^{-1}}{(1-0.4z^{-1})(1+0.3z^{-1})} \right] \left[ \frac{-1.5z^{-1} + 0.3z^{-2}}{1+0.25z^{-1}-0.06z^{-2}} \right].$$

A partial-fraction expansion of $Y(z)$ in $z^{-1}$ using the M-file \texttt{residue} yields

$$Y(z) = \frac{9.2045}{1+0.4z^{-1}} - \frac{3.15}{1-0.4z^{-1}} - \frac{5}{1+0.3z^{-1}} - \frac{1.0545}{1-0.15z^{-1}}, |z|>0.4.$$ 

Hence, from Table 6.1 we have $y[n] = \left(9.2045(-0.4)^{n}-3.15(0.4)^{n}-5(-0.3)^{n}-1.0545(0.15)^{n}\right)\mu[n].$

5. 3 points. Mitra 6.48 (a).

Solution to part (a):

Let the output of the predictor of Figure P6.4(a) be denoted by $E(z)$. Then analysis of this structure yields $E(z) = P(z)[U(z) + E(z)]$ and $U(z) = X(z) - E(z)$. From the first equation we have $E(z) = \frac{P(z)}{1-P(z)} U(z)$ which when substituted in the second equation yields

$$H(z) = \frac{U(z)}{X(z)} = 1 - P(z).$$

Analyzing Figure P6.3(b) we get $Y(z) = V(z) + P(z)Y(z)$ which leads to

$$G(z) = \frac{Y(z)}{V(z)} = \frac{1}{1-P(z)},$$

which is seen to be the inverse of $H(z)$.

Hence, for $P(z) = h_1z^{-1}$, we have $H(z) = 1 - h_1z^{-1}$ and $G(z) = \frac{1}{1-h_1z^{-1}}$. 
6. 3 points. Mitra 6.73

Solution:

(a) \( Y(z) = X(z) + \alpha X(z)z^{-M} \), therefore, \( H(z) = \frac{Y(z)}{X(z)} = 1 + \alpha z^{-M} \) and \( h[n] = \delta[n] + \alpha \delta[n-M] \).

(b) \( G(z) = \frac{1}{H(z)} = \frac{1}{1 + \alpha z^{-M}} = \sum_{k=0}^{\infty} (-1)^k \alpha^k z^{-kM} \) by long division. Therefore, \( g[n] = \sum_{k=0}^{\infty} (-\alpha)^k \delta[n-kM] \).

(c) The ROC of the causal \( g[n] \) is \(|z| > |(-\alpha)^{1/M}| \). As long as \(|(-\alpha)^{1/M}| < 1\), the ROC will contain the unit circle and the inverse system will be stable.

7. 4 points. Mitra 6.83

Solution: Since \( x[n] = \alpha^n \mu[n] \), we know from a simple table lookup that \( X(z) = \frac{1}{1-\alpha z^{-1}} \) with ROC \(|z| > |\alpha| \). We now let

\[
X[k] = X(z)|_{z = ze^{j2\pi k/N}} = \frac{1}{1 - \alpha e^{-j2\pi k/N}}
\]

Even though the notation looks like the DFT here, this isn’t the DFT of \( \{x[n]\} \) because \( \{x[n]\} \) is an infinite length sequence and the DFT is only used on finite-length sequences.

Now we save that result for a bit and form a periodic extension of \( \{x[n]\} \) as follows

\[
\tilde{x}[n] = \sum_{\ell=-\infty}^{\infty} x[n + \ell N]
\]

\[
= \sum_{\ell=-\infty}^{\infty} \alpha^{n+\ell N} \mu[n + \ell N]
\]

Let \( n = mN + p \) with \( m \in \mathbb{Z} \) and \( p = 0, 1, \ldots, N - 1 \). Then we can write

\[
\tilde{x}[mN + p] = \sum_{\ell=-\infty}^{\infty} \alpha^{mN+p+\ell N} \mu[mN + p + \ell N]
\]

\[
= \alpha^{mN+p} \sum_{\ell=-m}^{\infty} \alpha^{\ell N}
\]

\[
= \alpha^{mN+p} \frac{\alpha^{-mN}}{1 - \alpha^N}
\]

\[
= \frac{\alpha^p}{1 - \alpha^N}
\]

for \( n = mN + p \) and \( p = 0, 1, \ldots, N - 1 \). If you plot this, you will see a periodic sawtooth type of waveform. Now we take the DFT of one period of \( \tilde{x}[n] \). We can set \( m = 0 \) so that
\[ \tilde{x}[n] = \tilde{x}[p] \text{ for } n = 0, \ldots, N - 1 \text{ and write} \]

\[ \tilde{X}[k] = \sum_{p=0}^{N-1} \tilde{x}[p] e^{-j2\pi kp/N} \]

\[ = \frac{1}{1 - \alpha^N} \sum_{p=0}^{N-1} \alpha^p e^{-j2\pi kp/N} \]

\[ = \frac{1}{1 - \alpha^N} \sum_{p=0}^{N-1} \beta^p \]

\[ = \frac{1}{1 - \alpha^N} \sum_{p=0}^{N-1} \beta^p \]

\[ = \frac{1}{1 - \alpha^N} \frac{1 - \beta^N}{1 - \beta} \]

\[ = \frac{1}{1 - \alpha^N} \frac{1 - \alpha^N e^{-j2\pi k}}{1 - \alpha e^{-j2\pi k/N}} \]

\[ = \frac{1}{1 - \alpha e^{-j2\pi k/N}} \]

\[ = X[k] \]

since \( e^{-j2\pi k} = 1 \) for all integer \( k \). So this problem establishes an interesting relationship between the z-transform of an infinite length sequence sampled at specific values of \( z \) on the unit circle (corresponding to the DFT frequencies) and the actual DFT of one period of the periodically-extended sequence.