ECE503 Homework Assignment Number 5 Solution

1. 3 points. Mitra 7.25.

   Solution:

   $v[n] = x[-n]) \circledast h[n]$, and $u[n] = v[-n] = x[n] \circledast h[-n]$. Hence,

   $y[n] = (h[n] + h[-n]) \circledast x[n]$. Therefore, $G(e^{j\omega}) = H(e^{j\omega}) + H^*(e^{j\omega})$. Thus, the equivalent frequency response is real and has zero phase.

2. 3 points. Mitra 7.30.

   Solution:

   $$H_1(z) = 2.5\left(1 - 1.6z^{-1} + 2z^{-2}\right)\left(1 + z^{-1}\right)\left(1 + 1.6z^{-1} + z^{-2}\right)\left(1 - 0.8z^{-1} + 0.5z^{-2}\right)$$

   (a) $H_2(z) = 2.5\left(2 - 1.6z^{-1} + 1z^{-2}\right)\left(1 + z^{-1}\right)\left(1 + 1.6z^{-1} + z^{-2}\right)\left(1 - 0.8z^{-1} + 0.5z^{-2}\right)$
   $= 5 + 5z^{-1} + 0.4z^{-2} + 1.52z^{-3} + 4.17z^{-4} + 1.05z^{-5} - 0.75z^{-6} + 1.25z^{-7}$
   (b) $H_3(z) = 2.5\left(1 - 1.6z^{-1} + 2z^{-2}\right)\left(1 + z^{-1}\right)\left(1 + 1.6z^{-1} + z^{-2}\right)\left(0.5 - 0.8z^{-1} + 1z^{-2}\right)$
   $= 1.25 - 0.75z^{-1} + 1.05z^{-2} + 4.17z^{-3} + 1.52z^{-4} + 0.4z^{-5} + 5z^{-6} + 5z^{-7}$
   (c) $H_4(z) = H_1(z)$. There is only mixed-phase transfer function with the same magnitude response.
   (d) There are 3 total combinations having the same magnitude response. All 3 are listed here, and there are 0 others.

3. 3 points. Mitra 7.64 (a)

Solution to part (a):

(a) Using Eq. (7.81), we get $\beta = \cos(0.55\pi) = 0.4540$. Next, from Eq. (7.83), we get $\frac{2\alpha}{1+\alpha^2} = \cos(0.15\pi) = 0.8910$, or, equivalently, $0.8910\alpha^2 - 2\alpha + 0.8910 = 0$. Solution of this quadratic equation yields $\alpha = 1.6319$ and $\alpha = 0.6128$.

Substituting $\alpha = 1.6319$ and $\beta = 0.4540$ in Eq.(7.82) we arrive at the denominator polynomial of the transfer function $H_{BP}(z)$ as $D(z) = 1 - 1.1949z^{-1} + 1.6319z^{-2}$. Comparing with Eq. (7.161) we note $d_1 = -1.1949$ and $d_2 = 1.6319$. Since the condition of Eq. (7.164) is not satisfied, the corresponding $H_{BP}(z)$ is unstable.

Substituting $\alpha = 0.6128$ and $\beta = 0.4540$ in Eq.(7.82) we arrive at the denominator polynomial of the transfer function $H_{BP}(z)$ as $D(z) = 1 - 0.7322z^{-1} + 0.6128z^{-2}$. Comparing with Eq. (7.161) we note $d_1 = -0.7322$ and $d_2 = 0.6128$. Since the conditions of Eqs. (7.164) and (7.166) are satisfied, the corresponding $H_{BP}(z)$ is a stable transfer function. Hence, the desired transfer function is $H_{BP}(z) = \frac{0.1936(1 - z^{-2})}{1 - 0.7322z^{-1} + 0.6128z^{-2}}$.

4. 3 points. Mitra 7.76 (b)

Solution to part (b):

(b) $H_{BS}(z) = \frac{1}{16}(1 + z^{-2})(-1 + 6z^{-2} - z^{-4})$. Thus,

$$H_{BP}(z) = z^{-3} - \frac{1}{16}(1 + z^{-2})(-1 + 6z^{-2} - z^{-4}) = \frac{1}{16}(1 - 5z^{-2} + 16z^{-3} - 5z^{-4} + z^{-6}).$$

5. 3 points. Mitra 7.89 (a)

Solution to part (a):

(a) $H_a(z) = \frac{4.15 + 3.5z^{-1} + 4.15z^{-2}}{6.2 + 3.5z^{-1} + 2.1z^{-2}} = \frac{1}{2}[1 + \mathcal{A}(z)]$ where

$\mathcal{A}(z) = \frac{2.1 + 3.5z^{-1} + 6.2z^{-2}}{6.2 + 3.5z^{-1} + 2.1z^{-2}}$ is an allpass function. Hence, the power

complementary transfer function of $H_a(z)$ is given by

$$G_a(z) = \frac{1}{2}[1 - \mathcal{A}(z)] = \frac{1}{2}\left[1 - \frac{2.1 + 3.5z^{-1} + 6.2z^{-2}}{6.2 + 3.5z^{-1} + 2.1z^{-2}}\right] = \frac{2.05(1 - z^{-2})}{6.2 + 3.5z^{-1} + 2.1z^{-2}}.$$

6. 3 points. Mitra 7.90

Solution:

$H(z) = \dfrac{(2.2 + 5z^{-1})(1 - 3.1z^{-1})}{(1 + 0.81z^{-1})(1 - 0.62z^{-1})}$. In order to correct for magnitude distortion we require the transfer function $G(z)$ to satisfy the following property $\left|G(e^{j\omega})\right| = \dfrac{1}{\left|H(e^{j\omega})\right|}$. Hence, one possible solution is

$G_d(z) = \dfrac{1}{H(z)} = \dfrac{(1 + 0.81z^{-1})(1 - 0.62z^{-1})}{(2.2 + 5z^{-1})(1 - 3.1z^{-1})}$. Note that both poles are outside the unit circle, making $G_d(z)$ unstable.

To develop a stable transfer function with magnitude response same as $G_d(z)$, we multiply it with the stable allpass function $\dfrac{(2.2 + 5z^{-1})(1 - 3.1z^{-1})}{(5 + 2.2z^{-1})(-3.1 + z^{-1})}$ resulting in the transfer function $G(z) = \dfrac{(1 + 0.81z^{-1})(1 - 0.62z^{-1})}{(5 + 2.2z^{-1})(-3.1 + z^{-1})}$ which is the desired stable solution satisfying the condition $\left|G(e^{j\omega})\right|\left|H(e^{j\omega})\right| = 1$.

7. 4 points. Mitra 7.92 (a) and (b).

Solution to part (a):

We make use of Eqs. (7.135) and (7.136): $h[0] = \frac{y[0]}{x[0]}$ and

$$h[n] = \frac{y[n] - \sum_{k=0}^{n-1} h[k]x[n-k]}{x[0]}, \quad n \geq 1.$$

(a) $\{x_1[n]\} = \{2, \ -1, \ 3, \ 1\}$, $\{y_1[n]\} = \{4, \ -6, \ 14, \ -7, \ 7, \ 3\}$. Thus,

$h_1[0] = \frac{y_1[0]}{x_1[0]} = \frac{4}{2} = 2$, $h_1[1] = \frac{y_1[1] - h_1[0]x_1[1]}{x_1[0]} = \frac{-6 - 2 \times (-1)}{2} = -2$,

$h_1[2] = \frac{y_1[2] - h_1[0]x_1[2] - h_1[1]x_1[1]}{x_1[0]} = \frac{14 - 2 \times 3 - (-2) \times (-1)}{2} = 3.$

Hence $\{h_1[n]\} = \{2, \ -2, \ 3\}$.

Solution to part (b):

(b) $\{x_2[n]\} = \{3, \ 2, \ -1\}$, $\{y_2[n]\} = \{6, \ -5, \ -5 \ -1, \ -5 \ 2\}$.

$h_2[0] = \frac{y_2[0]}{x_2[0]} = \frac{6}{3} = 2$, $h_2[1] = \frac{y_2[1] - h_2[0]x_2[1]}{x_2[0]} = \frac{-5 - 2 \times 2}{3} = -3$,

$h_2[2] = \frac{y_2[2] - h_2[0]x_2[2] - h_2[1]x_2[1]}{x_2[0]} = \frac{-5 - 2 \times (-1) - (-3) \times 2}{3} = 1,$

$h_2[3] = \frac{y_2[3] - h_2[1]x_2[2] - h_2[2]x_2[1]}{x_2[0]} = \frac{-1 - (-3) \times (-1) - 1 \times 2}{3} = 2.$

Hence $\{h_2[n]\} = \{2, \ -3, \ 1, \ -2\}$.

8. 3 points. Mitra M7.9.

Solution: We follow the procedure in 7.4.2 for the lowpass filter. Given $\omega_c = 0.2\pi$, we can use (7.78b) to solve $\alpha = \frac{1-\sin\omega_c}{\cos\omega_c} = 0.5095$ which we plug in to the scaled transfer function

$$H_{LP}(z) = \frac{1-\alpha}{2}\frac{1+z^{-1}}{1-\alpha z^{-1}}.$$

This same α can be used for the high pass filter

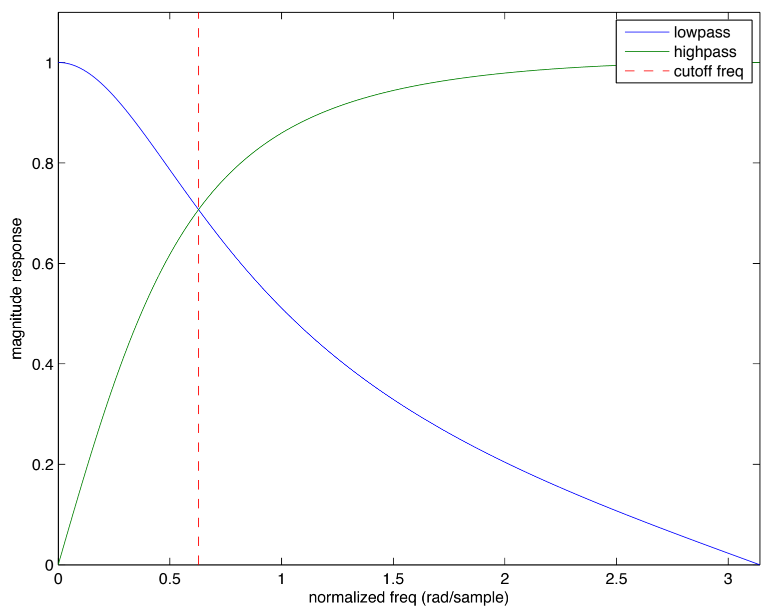
$$H_{HP}(z) = \frac{1+\alpha}{2}\frac{1-z^{-1}}{1-\alpha z^{-1}}.$$

The following Matlab code plots the magnitude response of both filters on the same plot and shows these transfer functions are allpass complementary and power complementary.

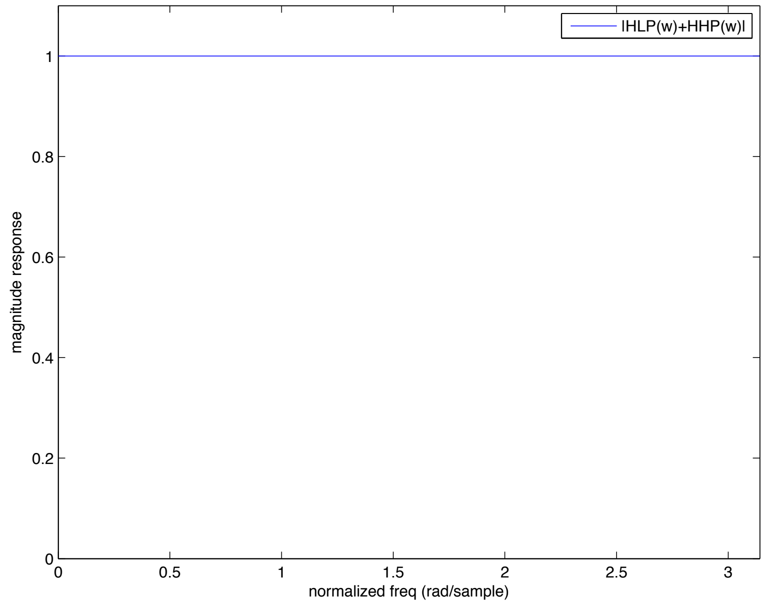
```
% Mitra M7.9
% DRB 08-Feb-2012

% --------------------------------------------
% plot magnitude responses
% --------------------------------------------
wc = 0.2*pi;
alpha = (1-sin(wc))/cos(wc);

blp = (1-alpha)/2*[1 1];
alp = [1 -alpha];

bhp = (1+alpha)/2*[1 -1];
ahp = [1 -alpha];

w = 0:0.001:pi;     % normalized frequency scale
hlp = freqz(blp,alp,w);
hhp = freqz(bhp,ahp,w);
figure(1)
plot(w,abs(hlp),w,abs(hhp),[0.2 0.2]*pi,[0 1.1],'--');
legend('lowpass','highpass','cutoff freq');
xlabel('normalized freq (rad/sample)');
ylabel('magnitude response');
axis([0 pi 0 1.1]);

% --------------------------------------------
% show allpass complementary
% --------------------------------------------
bsum = blp+bhp;         % add numerators
asum = alp;             % same denominator for LP and HP

hsum = freqz(bsum,asum,w);
figure(2)
plot(w,abs(hsum));
legend('|HLP(w)+HHP(w)|');
```

```
xlabel('normalized freq (rad/sample)');
ylabel('magnitude response');
axis([0 pi 0 1.1]);

% ---------------------------------------------
% show power complementary
% ---------------------------------------------
pc = abs(hlp).^2 + abs(hhp).^2;  % sum squared magnitude responses
figure(3)
plot(w,pc);
legend('|HLP(w)|^2+|HHP(w)|^2');
xlabel('normalized freq (rad/sample)');
ylabel('total squared magnitude response');
axis([0 pi 0 1.1]);
```

The magnitude response plot is given below.



The complementary properties are easy to verify analytically, but the following plot confirms these filters are allpass complementary.



The following plot confirms these filters are power complementary.