1. 3 points. Mitra 8.4.

The structure with internal variables is shown below. Analysis of this structure yields

\[
U(z) = KX(z) + d_2 z^{-1} V(z), \quad V(z) = U(z) - d_1 z^{-1} V(z), \quad Y(z) = d_1 z^{-1} V(z) - z^{-2} V(z) - d_1 V(z).
\]

Eliminating the internal variables we arrive at

\[
H(z) = \frac{Y(z)}{X(z)} = \frac{K(-d_2 + d_1 z^{-1} + z^{-2})}{1 + d_1 z^{-1} - d_2 z^{-2}}.
\]

(a) Since the transfer function is second-order, the structure is non-canonic.

(b) \(H(e^{j\omega}) = \frac{K(-d_2 + d_1 + 1)}{1 + d_1 - d_2} = K\). Hence the structure has a unity gain at \(\omega = 0\) if \(K = 1\).

(c) \(H(e^{j\pi}) = \frac{K(-d_2 - d_1 + 1)}{1 - d_1 - d_2} = K\). Hence the structure has a unity gain at \(\omega = \pi\) if \(K = 1\).

(d) If we let \(D(z) = 1 + d_1 z^{-1} - d_2 z^{-2}\), then \(H(z) = \frac{Kz^{-2} D(z^{-1})}{D(z)}\). Now,

\[
H(z)H(z^{-1}) = \frac{Kz^{-2} D(z^{-1})}{D(z)} \cdot \frac{Kz^2 D(z)}{D(z^{-1})} = K^2. \text{ This implies}
\]

\[
\left|H(e^{j\omega})\right|^2 = H(z)H(z^{-1})\bigg|_{z=e^{j\omega}} = K^2, \text{ or in other words the transfer function has a constant magnitude for all values of } \omega.
\]

2. 4 points. Mitra 8.13 (a) and (c).

(a) \(H(z) = (1 + 0.4z^{-1})^4(1 - 0.2z^{-1})^2\)

\[= 1 + 1.2z^{-1} + 0.36z^{-2} - 0.64z^{-3} - 0.0384z^{-4} + 0.001z^{-5}.\]

A direct form realization of \(H(z)\) is shown below:
The transposed form of the above structure yields another direct form realization as indicated below:

(c) A realization in the form of cascade of three second-order sections is shown below:

3. 3 points. Mitra 8.19.

\[ G(z) = z^{-N/2} - H(z) \]. A canonic realization of both \( G(z) \) and \( H(z) \) is shown below for \( N = 6 \).
4. 5 points. Mitra 8.33 (a)—(e). Note the implicit assumption in part (e) that the system is relaxed.

(a) \( Y(z) = \frac{1}{1-0.3z^{-1}} - \frac{0.5z^{-1}}{1-0.3z^{-1}} = \frac{1-0.5z^{-1}}{1-0.3z^{-1}}, X(z) = \frac{1}{1-0.7z^{-1}}. \) Thus,

\[
H(z) = \frac{Y(z)}{X(z)} = \frac{(1-0.5z^{-1})(1-0.7z^{-1})}{1-0.3z^{-1}} = \frac{1-1.2z^{-1} + 0.35z^{-2}}{1-0.3z^{-1}}.
\]

(b) \( y[n] = x[n] - 1.2x[n-1] + 0.35x[n-2] + 0.3y[n-1]. \)

(c) A Direct Form II realization is shown below:

(d) A partial-fraction expansion of \( H(z) \) in \( z^{-1} \) using \( \text{residue}z \) is given by

\[
H(z) = 0.1111z^{-1} - 1.1667 + \frac{0.8889}{1-0.3z^{-1}} \]

whose realization yields the Parallel Form I structure shown below.

(e) The inverse \( z \)-transform of \( H(z) \) yields

\[
h[n] = 0.1111[n-1] - 1.1667[n] + 0.8889(0.3)^n \mu[n].
\]
5. 6 points total. Suppose you have a system described by the difference equation

\[ y[n] = \frac{1}{2}(x[n] + x[n - 1]) + 0.9y[n - 1] \]

(a) 1 point. Write the transfer function for this system (including the ROC).

**Solution:** We can take \( z \)-transforms of both sides to write

\[ Y(z) = \frac{1}{2}(X(z) + z^{-1}X(z)) + 0.9z^{-1}Y(z) \]

and collect terms to write

\[ H(z) = \frac{Y(z)}{X(z)} = \frac{\frac{1}{2}(1 + z^{-1})}{1 - 0.9z^{-1}} \]

Since the original difference equation specifies a causal system, then the ROC must extend outward from the largest magnitude pole. Hence the ROC here is \( |z| > 0.9 \).

(b) 1 point. Use Matlab to plot the magnitude and phase response of this system.

**Solution:** Here is what I would do:

```matlab
b = [0.5 0.5];
a = [1 -0.9];
freqz(b,a,1024)
```

Note the lowpass nature of this transfer function, which is what we would expect with a zero at \( z = -1 \) and a pole at \( z = 0.9 \). Also note that Matlab is only plotting the magnitude and phase on \( \omega \in [0, \pi] \), which is sufficient to understand the whole spectrum due to the symmetry and periodicity properties of the DTFT of a real sequence.
(c) 1 point. Draw a direct form II realization of this system.

**Solution:**

Check:

\[ u[n] = x[n] + 0.9u[n-1] \iff U(z) = \frac{1}{1-0.9z^{-1}}X(z) \]

\[ y[n] = 0.5u[n] + 0.5u[n-1] \iff Y(z) = \frac{1}{2}(1+z^{-1})U(z) \]

Put these together to write

\[ Y(z) = \frac{1}{2}(1+z^{-1})U(z) = \frac{1}{2}(1+z^{-1})(1-0.9z^{-1})X(z) \]

which is what we wanted.

(d) 3 points. Compute the step response of this system given \( y[-1] = 5 \).

**Solution:** There are several ways to do this, but here is what I would do. I would first compute the zero-state response using the transfer function. Note that \( X(z) = \frac{1}{1-z^{-1}} \) with ROC \( |z| > 1 \). Hence

\[ Y(z) = \frac{1}{2}(1+z^{-1})U(z) = \frac{1}{2}(1+z^{-1})(1-0.9z^{-1})X(z) \]

with ROC \( |z| > 1 \) since \( y[n] \) must be a right-sided sequence. We can use the usual PFE techniques to find \( a \) and \( b \). I get \( a = -9.5 \) and \( b = 10 \). Since the ROC extends outward, we know the inverse \( z \)-transforms must be right sided sequences and we can find the inverse \( z \)-transform by inspection / table lookup as

\[ y_{ZS}[n] = -9.5 \cdot 0.9^n \mu[n] + 10 \mu[n] \]

This can be confirmed in Matlab with the `step` function or the `filter` function, e.g. \( y = \text{filter}([0.5 0.5],[1 -0.9],\text{ones}(1,10)) \). This is only the zero-state response, however. Our system is not relaxed. To compute the total response, we also need the zero-input response of the system. Looking back at the original difference equation, but setting the input to zero, we can write

\[ y[n] - 0.9y[n-1] = 0 \]

which will have a complementary solution of the form \( y_c[n] = \alpha 0.9^n \). We use the initial condition \( y[-1] = 5 \) to determine \( \alpha \). I get \( \alpha = \frac{5}{0.9^{-1}} = 4.5 \). Hence, the zero-input response is

\[ y_{ZI}[n] = 4.5 \cdot 0.9^n \]

for \( n \geq 0 \). This agrees with \( y[0] = 0.9y[-1] = 0.9 \cdot 5 = 4.5 \) computed directly from the difference equation. We can also confirm this in Matlab, either by writing a for
loop, applying the initial condition, and computing the zero-input recursion directly, or by following the approach from lecture 2 to determine the internal initial condition $z_1[-1] = y[0] = 4.5$ and using the Matlab command $y = \text{filter}([0.5 0.5],[1 -0.9],\text{zeros}(1,10),4.5)$. This agrees with the zero-input response computed above. The total solution for $n \geq 0$ is then

$$y[n] = y_{ZS}[n] + y_{ZI}[n]$$

$$= -5 \cdot 0.9^n \mu[n] + 10\mu[n].$$
6. 2 points. Mitra 5.50 (b). Hint: The midterm exam problem 3 solution might be helpful.

(b) Now we can write \( g[n] \) as: 
\[
g[n] = \frac{1}{2} \left( x[n] + (-1)^n x[n] \right) = \frac{1}{2} \left( x[n] + W_N^{-(N/2)n} x[n] \right).
\]

Using circular frequency shifting theorem of the DFT given in Table 5.3, we get:
\[
G[k] = \text{DFT}(g[n]) = \frac{1}{2} \left( X[k] + X[k - N/2] \right).
\]

7. 2 points. Mitra 5.52 (c).

(c) Using the modulation property of the DFT given in Table 5.3:
\[
y[n] = \text{IDFT}\{Y[k]\} = N \cdot x[n] \cdot x[n] = N \cdot x^2[n].
\]