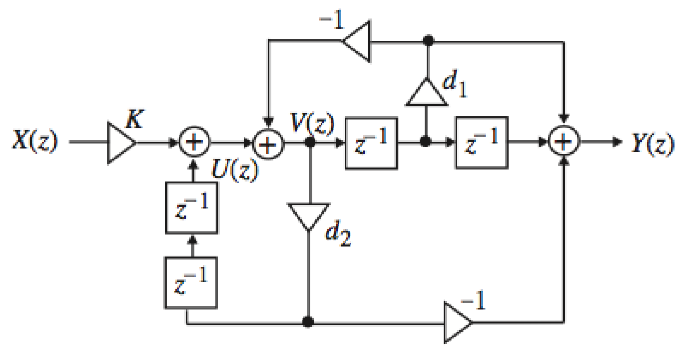
ECE503 Homework Assignment Number 6 Solution

1. 3 points. Mitra 8.4.

The structure with internal variables is shown below. Analysis of this structure yields



$U(z) = KX(z) + d_2 z^{-1} V(z)$, $V(z) = U(z) - d_1 z^{-1} V(z)$, $Y(z) = d_1 z^{-1} V(z) - z^{-2} V(z) - d_1 V(z)$.
Eliminating the internal variables we arrive at

$$H(z) = \frac{Y(z)}{X(z)} = \frac{K(-d_2 + d_1 z^{-1} + z^{-2})}{1 + +d_1 z^{-1} - d_2 z^{-2}}.$$

(a) Since the transfer function is second-order, the structure is non-canonic.

(b) $H(e^{j0}) = \dfrac{K(-d_2 + d_1 + 1)}{1 + d_1 - d_2} = K$. Hence the structure has a unity gain at $\omega = 0$ if $K = 1$.

(c) $H(e^{j\pi}) = \dfrac{K(-d_2 - d_1 + 1)}{1 - d_1 - d_2} = K$. Hence the structure has a unity gain at $\omega = \pi$ if $K = 1$.

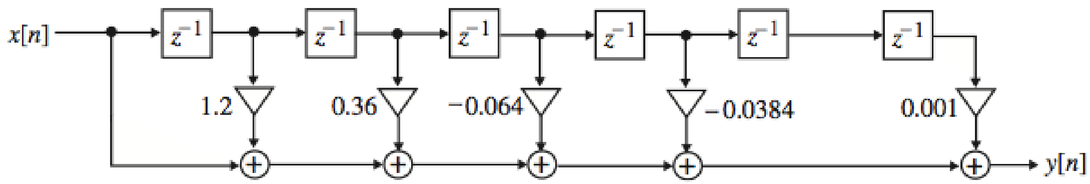
(d) If we let $D(z) = 1 + d_1 z^{-1} - d_2 z^{-2}$, then $H(z) = \dfrac{K z^{-2} D(z^{-1})}{D(z)}$. Now,

$H(z)H(z^{-1}) = \dfrac{K z^{-2} D(z^{-1})}{D(z)} \cdot \dfrac{K z^{2} D(z)}{D(z^{-1})} = K^2$. This implies

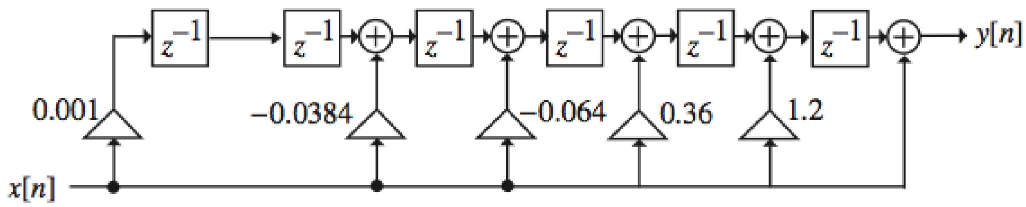
$\left|H(e^{j\omega})\right|^2 = H(z)H(z^{-1})\Big|_{z=e^{j\omega}} = K^2$, or in other words the transfer function has a constant magnitude for all values of ω.

2. 4 points. Mitra 8.13 (a) and (c).

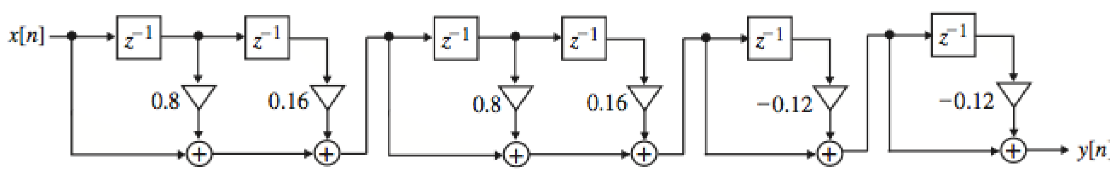
(a) $H(z) = (1 + 0.4z^{-1})^4 (1 - 0.2z^{-1})^2$
$= 1 + 1.2z^{-1} + 0.36z^{-2} - 0.64z^{-3} - 0.0384z^{-4} + 0.001z^{-6}$. A direct form realization of $H(z)$ is shown below:



The transposed form of the above structure yields another direct form realization as indicated below:

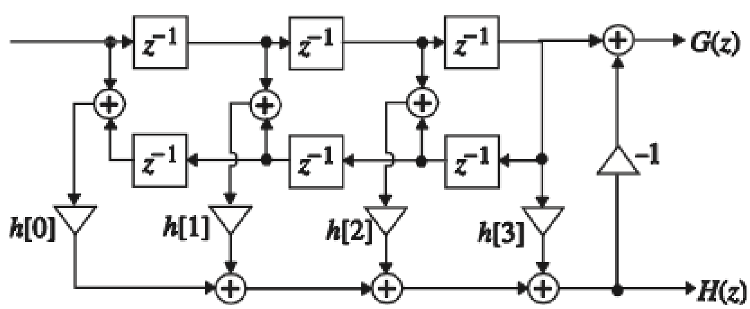


(c) A realization in the form of cascade of three second-order sections is shown below:



3. 3 points. Mitra 8.19.

$G(z) = z^{-N/2} - H(z)$. A canonic realization of both $G(z)$ and $H(z)$ is shown below for $N = 6$.



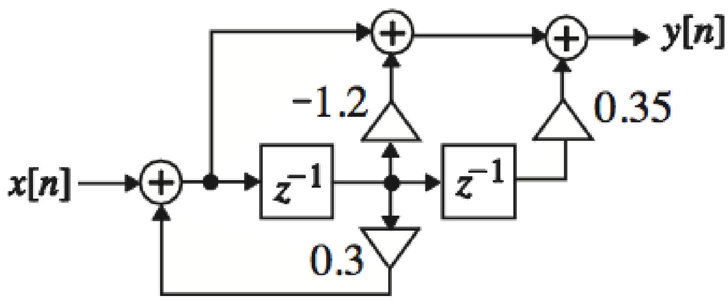
4. 5 points. Mitra 8.33 (a)—(e). Note the implicit assumption in part (e) that the system is relaxed.

(a) $Y(z) = \frac{1}{1-0.3z^{-1}} - \frac{0.5z^{-1}}{1-0.3z^{-1}} = \frac{1-0.5z^{-1}}{1-0.3z^{-1}}$, $X(z) = \frac{1}{1-0.7z^{-1}}$. Thus,

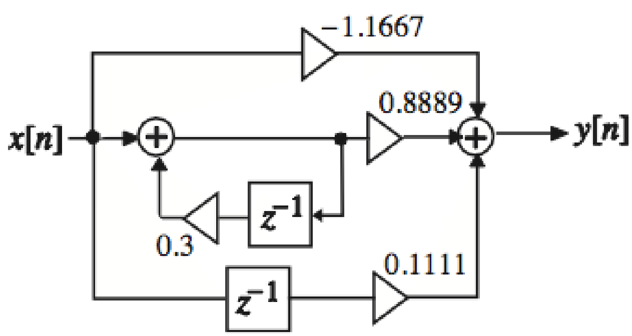
$$H(z) = \frac{Y(z)}{X(z)} = \frac{(1-0.5z^{-1})(1-0.7z^{-1})}{1-0.3z^{-1}} = \frac{1-1.2z^{-1}+0.35z^{-2}}{1-0.3z^{-1}}.$$

(b) $y[n] = x[n] - 1.2x[n-1] + 0.35x[n-2] + 0.3y[n-1]$.

(c) A Direct Form II realization is shown below:



(d) A partial-fraction expansion of $H(z)$ in z^{-1} using `residuez` is given by $H(z) = 0.1111z^{-1} - 1.1667 + \frac{0.8889}{1-0.3z^{-1}}$ whose realization yields the Parallel Form I structure shown below.



(e) The inverse z-transform of $H(z)$ yields
$h[n] = 0.1111[n-1] - 1.1667[n] + 0.8889(0.3)^n \mu[n]$.

5. 6 points total. Suppose you have a system described by the difference equation

$$y[n] = \frac{1}{2}(x[n] + x[n-1]) + 0.9y[n-1]$$

(a) 1 point. Write the transfer function for this system (including the ROC).

Solution: We can take z-transforms of both sides to write

$$Y(z) = \frac{1}{2}(X(z) + z^{-1}X(z)) + 0.9z^{-1}Y(z)$$

and collect terms to write

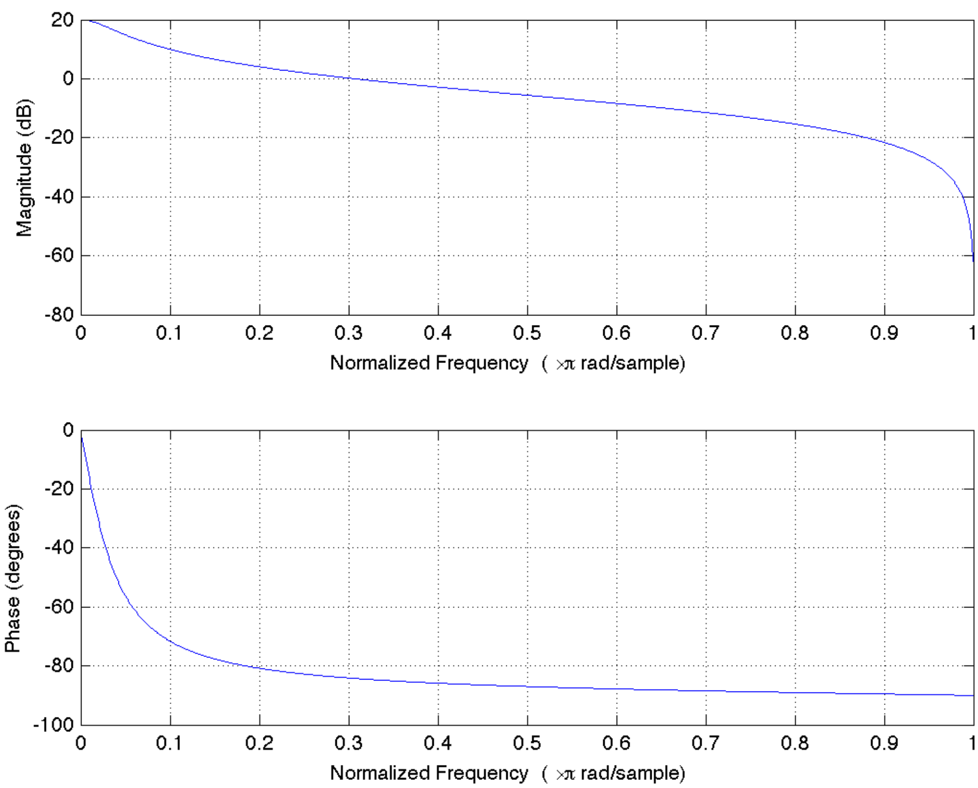
$$H(z) = \frac{Y(z)}{X(z)} = \frac{\frac{1}{2}(1 + z^{-1})}{1 - 0.9z^{-1}}$$

Since the original difference equation specifies a causal system, then the ROC must extend outward from the largest magnitude pole. Hence the ROC here is $|z| > 0.9$.

(b) 1 point. Use Matlab to plot the magnitude and phase response of this system.

Solution: Here is what I would do:

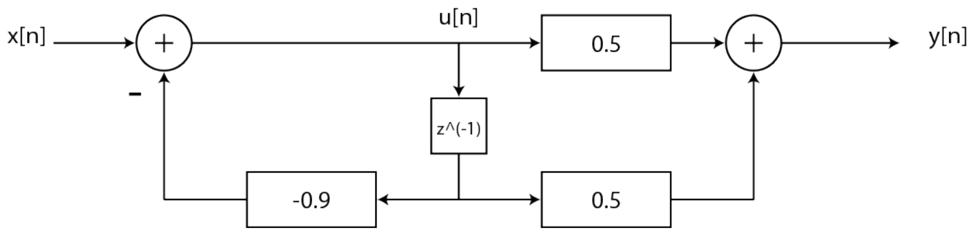
```
b = [0.5 0.5];
a = [1 -0.9];
freqz(b,a,1024)
```



Note the lowpass nature of this transfer function, which is what we would expect with a zero at $z = -1$ and a pole at $z = 0.9$. Also note that Matlab is only plotting the magnitude and phase on $\omega \in [0, \pi]$, which is sufficient to understand the whole spectrum due to the symmetry and periodicity properties of the DTFT of a real sequence.

(c) 1 point. Draw a direct form II realization of this system.

Solution:



Check:

$$u[n] = x[n] + 0.9u[n-1] \qquad \Leftrightarrow \qquad U(z) = \frac{1}{1-0.9z^{-1}}X(z)$$

$$y[n] = 0.5u[n] + 0.5u[n-1] \qquad \Leftrightarrow \qquad Y(z) = \frac{1}{2}(1+z^{-1})U(z)$$

Put these together to write

$$Y(z) = \frac{1}{2}(1+z^{-1})U(z) = \frac{\frac{1}{2}(1+z^{-1})}{1-0.9z^{-1}}X(z)$$

which is what we wanted.

(d) 3 points. Compute the step response of this system given $y[-1] = 5$.

Solution: There are several ways to do this, but here is what I would do. I would first compute the zero-state response using the transfer function. Note that $X(z) = \frac{1}{1-z^{-1}}$ with ROC $|z| > 1$. Hence

$$Y(z) = \frac{\frac{1}{2}(1+z^{-1})}{(1-0.9z^{-1})(1-z^{-1})} = \frac{a}{1-0.9z^{-1}} + \frac{b}{1-z^{-1}}$$

with ROC $|z| > 1$ since $y[n]$ must be a right-sided sequence. We can use the usual PFE techniques to find a and b. I get $a = -9.5$ and $b = 10$. Since the ROC extends outward, we know the inverse z transforms must be right sided sequences and we can find the inverse z-transform by inspection / table lookup as

$$y_{ZS}[n] = -9.5 \cdot 0.9^n \mu[n] + 10\mu[n].$$

This can be confirmed in Matlab with the `step` function or the `filter` function, e.g. `y = filter([0.5 0.5],[1 -0.9],ones(1,10)`. This is only the zero-state response, however. Our system is not relaxed. To compute the total response, we also need the zero-input response of the system. Looking back at the original difference equation, but setting the input to zero, we can write

$$y[n] - 0.9y[n-1] = 0$$

which will have a complementary solution of the form $y_c[n] = \alpha 0.9^n$. We use the initial condition $y[-1] = 5$ to determine α. I get $\alpha = \frac{5}{0.9^{-1}} = 4.5$. Hence, the zero-input response is

$$y_{ZI}[n] = 4.5 \cdot 0.9^n$$

for $n \geq 0$. This agrees with $y[0] = 0.9y[-1] = 0.9 \cdot 5 = 4.5$ computed directly from the difference equation. We can also confirm this in Matlab, either by writing a for loop, applying the initial condition, and computing the zero-input recursion directly, or by following the approach from lecture 2 to determine the internal initial condition $z_1[-1] = y[0] = 4.5$ and using the Matlab command `y = filter([0.5 0.5],[1 -0.9],zeros(1,10),4.5)`. This agrees with the zero-input response computed above. The total solution for $n \geq 0$ is then

$$\begin{aligned} y[n] &= y_{ZS}[n] + y_{ZI}[n] \\ &= -5 \cdot 0.9^n \mu[n] + 10\mu[n]. \end{aligned}$$

6. 2 points. Mitra 5.50 (b). Hint: The midterm exam problem 3 solution might be helpful.

(b) Now we can write $g[n]$ as: $g[n] = \frac{1}{2}\left(x[n] + (-1)^n x[n]\right) = \frac{1}{2}\left(x[n] + W_N^{-(N/2)n} x[n]\right)$.

Using circular frequency shifting theorem of the DFT given in Table 5.3, we get:
$G[k] = \text{DFT}(g[n]) = \frac{1}{2}\left(X[k] + X[\langle k - \frac{N}{2}\rangle_N]\right)$.

7. 2 points. Mitra 5.52 (c).

(c) Using the modulation property of the DFT given in Table 5.3:

$$y[n] = \text{IDFT}\{Y[k]\} = N \cdot x[n] \cdot x[n] = N \cdot x^2[n].$$