ECE503 Homework Assignment Number 7 Solution

Due by 8:50pm on Monday 26-Mar-2012

1. 3 points. Suppose a discrete-time signal

$$x[n] = \begin{cases} 0.9^n & n = 0, \dots, 9 \\ 0 & \text{otherwise.} \end{cases}$$

is sent through an ideal reconstruction filter with sampling period $T = \frac{1}{10}$ seconds to generate a continuous-time signal $x(t)$.

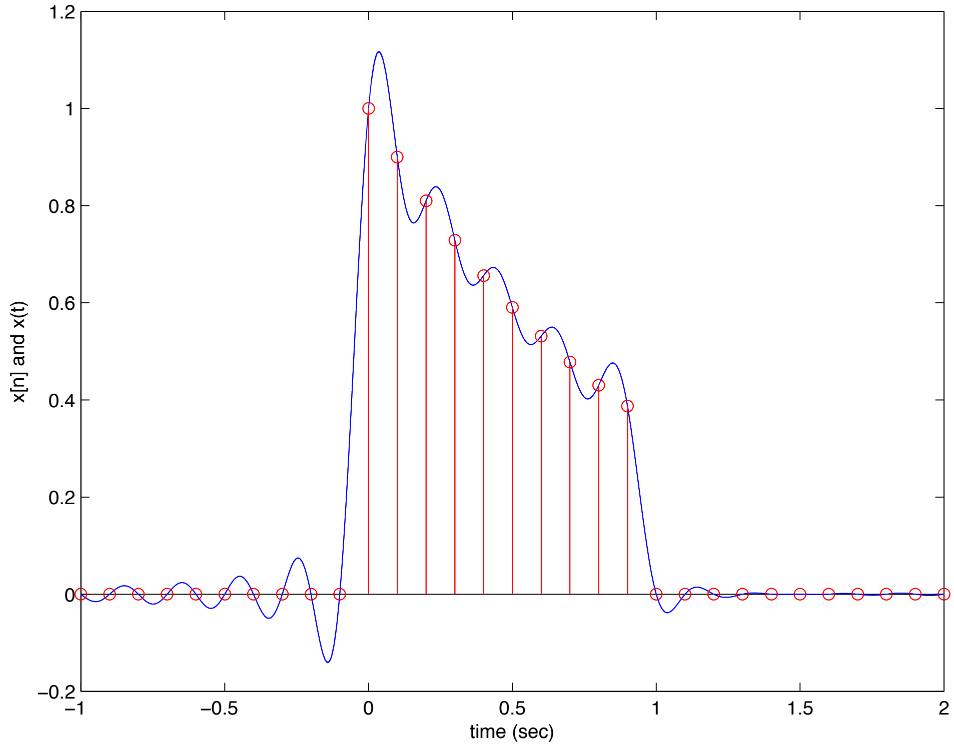
   (a) Note that $x[n] = 0$ for all $n < 0$. Does $x(t) = 0$ for all $t < 0$? Why or why not?
   Solution: The ideal reconstruction formula states

$$\begin{aligned} x(t) &= \sum_{n=-\infty}^{\infty} x[n] \frac{\sin(\pi(t-nT)/T)}{\pi(t-nT)/T} \\ &= \sum_{n=0}^{9} 0.9^n \frac{\sin(10\pi(t-n/10))}{10\pi(t-n/10)}. \end{aligned}$$

   To see that $x(t)$ is not equal to zero for all $t < 0$, let's pick, for example, $t = -1/20$. Then

$$\begin{aligned} x(t=-1/20) &= \sum_{n=0}^{9} 0.9^n \frac{\sin(10\pi(-1/20-n/10))}{10\pi(-1/20-n/10)} \\ &= \sum_{n=0}^{9} 0.9^n \frac{\sin(-\pi/2-n\pi)}{-\pi/2-n\pi} \\ &= \sum_{n=0}^{9} 0.9^n \frac{\sin(\pi/2+n\pi)}{\pi/2+n\pi} \\ &\approx 0.5036 \end{aligned}$$

   The following figure shows $x(t)$ and $x[n]$ as related by the ideal interpolation formula when $T = \frac{1}{10}$ seconds.




```
tt = -1:0.0001:2;
T = 1/10;
n = 0:9;
alpha = 0.9;
x = zeros(1,length(tt));
i1 = 0;
for t = tt,
    i1 = i1+1;
    x(i1) = sum((alpha.^n).*sin(pi*(t-n*T)/T)./(pi*(t-n*T)/T));
end
plot(tt,x);
hold on
stem(-1:T:2,[zeros(1,10) 0.9.^n zeros(1,11)],'r');
xlabel('time (sec)');
ylabel('x[n] and x(t)')
```

(b) Determine the value of $x(t)$ at time $t = 0.5$ seconds.
Solution: The ideal reconstruction formula states

$$
\begin{aligned}
x(t) &= \sum_{n=-\infty}^{\infty} x[n] \frac{\sin(\pi(t-nT)/T)}{\pi(t-nT)/T} \\
&= \sum_{n=0}^{9} 0.9^n \frac{\sin(10\pi(0.5-n/10))}{10\pi(0.5-n/10)} \\
&= \sum_{n=0}^{9} 0.9^n \frac{\sin(5\pi-n\pi)}{5\pi-n\pi}
\end{aligned}
$$

Note

$$\frac{\sin(5\pi - n\pi)}{5\pi - n\pi} = \begin{cases} 1 & n = 5 \\ 0 & n \neq 5. \end{cases}$$

So

$$\begin{aligned} x(t) &= \sum_{n=0}^{\infty} 0.9^n \delta[n-5] \\ &= 0.9^5 \approx 0.5905 \end{aligned}$$

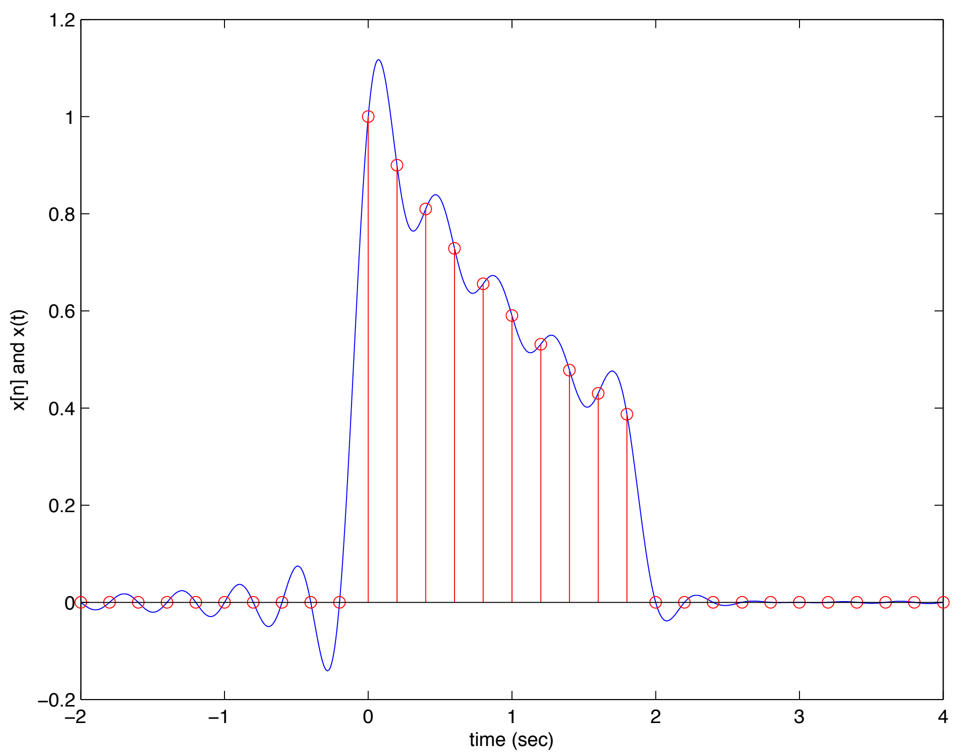
Note that $x(t = 0.5) = x[n = 5]$ since this time falls directly on a sampling instant.

(c) Now suppose the sampling period $T = \frac{1}{5}$ seconds. Determine the value of $x(t)$ at time $t = 0.5$ seconds.

Solution: The ideal reconstruction formula states

$$\begin{aligned} x(t) &= \sum_{n=-\infty}^{\infty} x[n] \frac{\sin(\pi(t - nT)/T)}{\pi(t - nT)/T} \\ &= \sum_{n=0}^{9} 0.9^n \frac{\sin(5\pi(0.5 - n/5))}{5\pi(0.5 - n/5)} \\ &= \sum_{n=0}^{9} 0.9^n \frac{\sin(2.5\pi - n\pi)}{2.5\pi - n\pi} \\ &\approx 0.8316 \end{aligned}$$

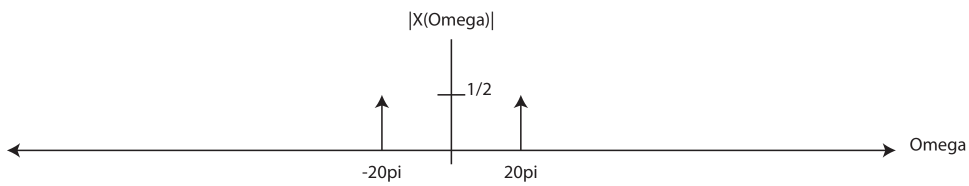
where the final result was computed in Matlab. Note this time instant falls between two samples, so you have to compute the full sum.

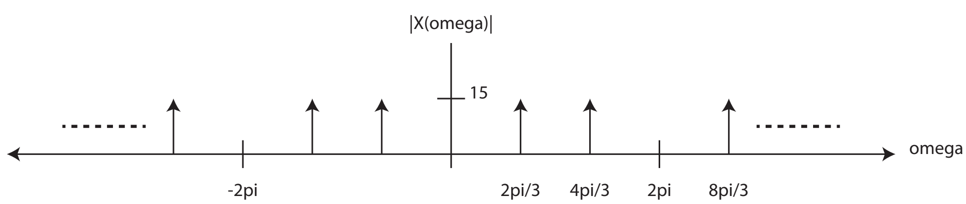


```
tt = -2:0.0001:4;
T = 1/5;
n = 0:9;
alpha = 0.9;
x = zeros(1,length(tt));
i1 = 0;
for t = tt,
    i1 = i1+1;
    x(i1) = sum((alpha.^n).*sin(pi*(t-n*T)/T)./(pi*(t-n*T)/T));
end
plot(tt,x);
hold on
stem(-2:T:4,[zeros(1,10) 0.9.^n zeros(1,11)],'r');
xlabel('time (sec)');
ylabel('x[n] and x(t)')
```

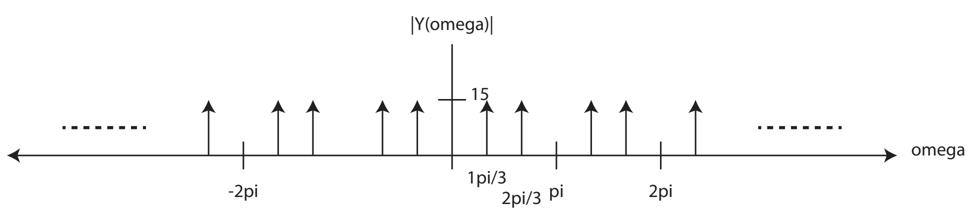
2. 4 points. Suppose you have a real-valued continuous-time signal $x(t) = \cos(2\pi \cdot 10 \cdot t)$ and this signal is ideally sampled at frequency $F_T = 30$ Hertz to to generate a discrete-time signal $x[n]$.

   (a) Sketch the magnitude of $X(\Omega)$ and the magnitude of $X(\omega)$, explicitly showing any periodicity in the spectra. Is there aliasing?
   Solution: There is no aliasing. The spectra are sketched below.



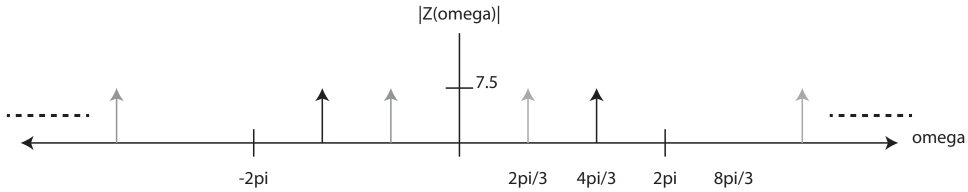


   (b) Now suppose $x[n]$ is upsampled by a factor of two, resulting in $y[n]$. Sketch the magnitude of $Y(\omega)$, explicitly showing any periodicity in the spectrum.
   Solution: There is still no aliasing here. The spectrum is sketched below. Note the additional tones that appear because of the up sampling. These tones could be attenuated/removed, if desired, with an interpolation filter.

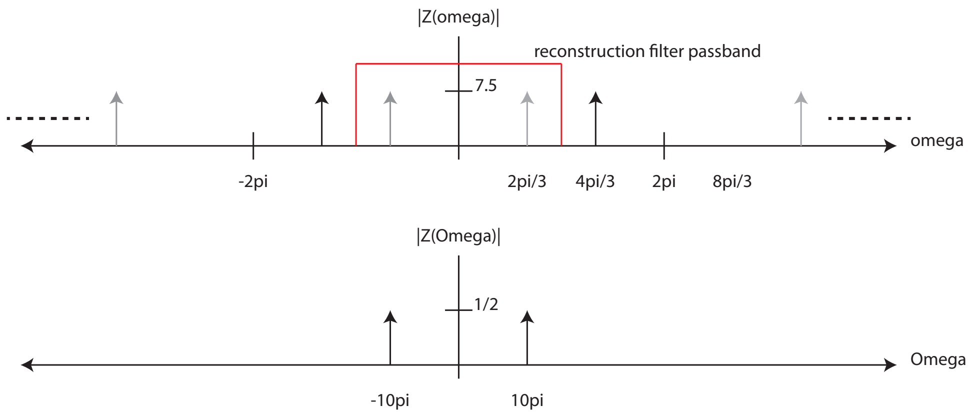


   (c) Now suppose $x[n]$ is downsampled by a factor of two, resulting in $z[n]$. Sketch the magnitude of $Z(\omega)$, explicitly showing any periodicity in the spectrum.

Solution: Now there is aliasing. The spectrum is sketched below. The images at $\pm 2\pi$ are shown in gray.



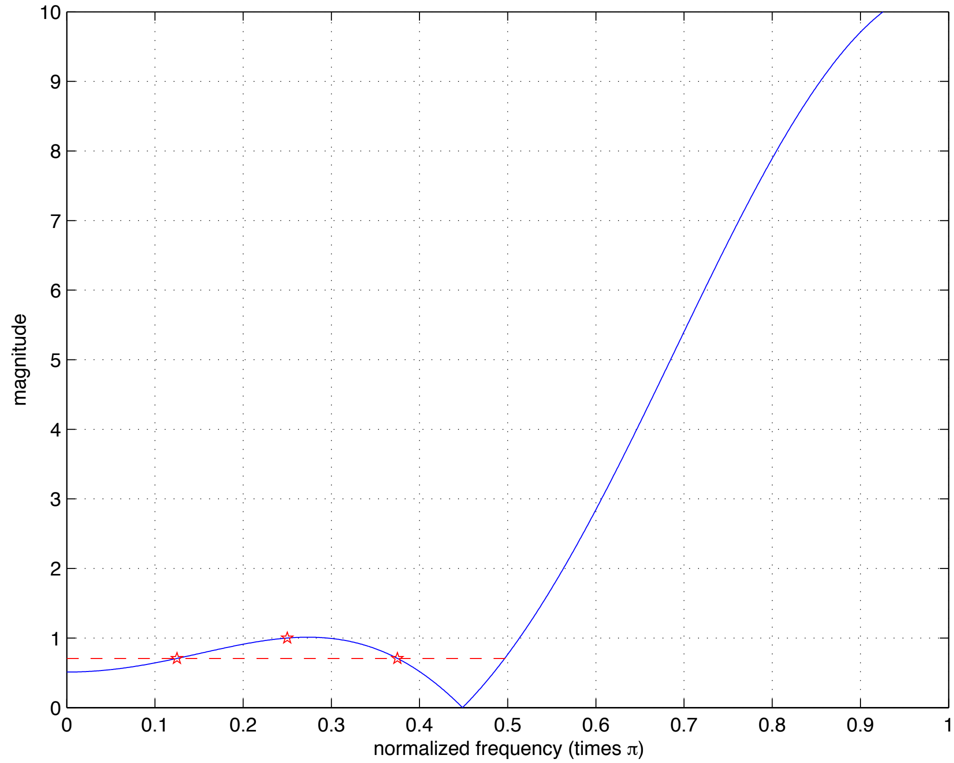
(d) Now suppose $z[n]$ is sent to an ideal reconstruction filter to generate $z(t)$. Note this ideal reconstruction filter will use a period of $T = \frac{1}{15}$ seconds because the sampling rate of $z[n]$ is 15 Hertz. Can you find a closed-form expression for $z(t)$?
Solution: In this case, reconstruction is easily analyzed in the frequency domain. The ideal reconstruction filter will only pass normalized frequencies between $-\pi$ and $+\pi$, hence we can sketch the spectra of $Z(\omega)$ and $Z(\Omega)$ as



It is clear that $z(t) = \cos(2\pi \cdot 5 \cdot t)$. Aliasing in the downsampled signal caused the ideal reconstruction filter output to be different than the original time domain signal $x(t)$.

3. 4 points total. Suppose you wish to design a "bandpass" filter that has unity magnitude at $\omega_1 = \frac{\pi}{4}$ and has magnitude $\frac{1}{\sqrt{2}}$ at $\omega_1 = \frac{\pi}{4} \pm \omega_0$.

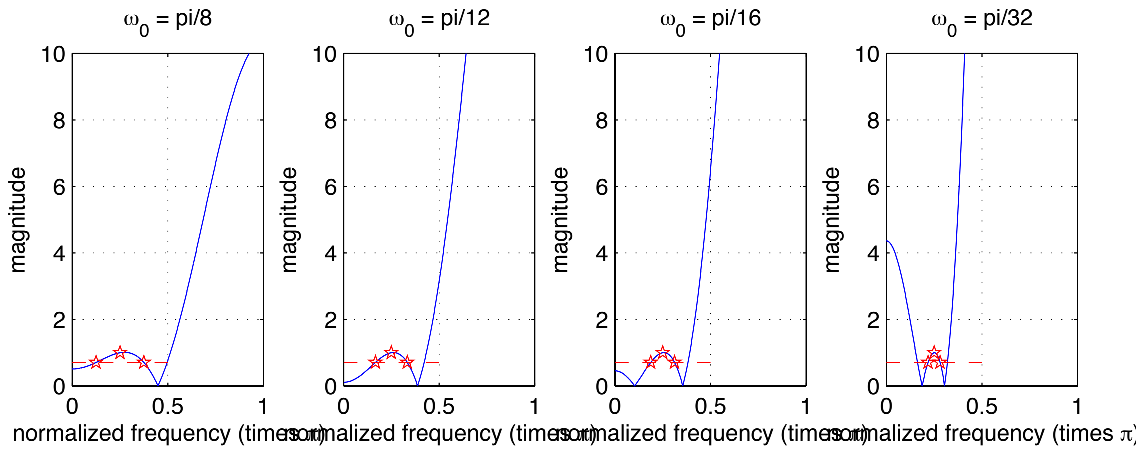
(a) 2 points. Design a filter that meets these specifications when $\omega_0 = \pi/8$. Use Matlab to plot the magnitude response and confirm it agrees with the specifications. Is this a good bandpass filter? Why or why not?
Solution:

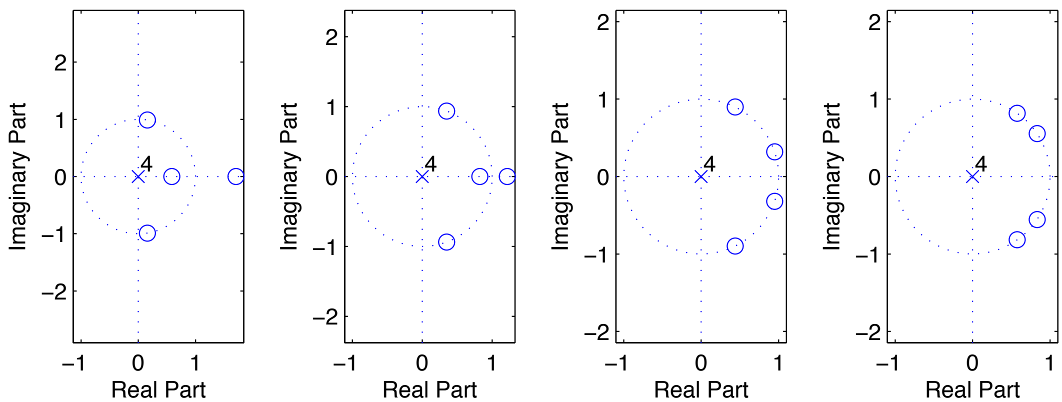



```
w1 = pi/4;
w0 = pi/8;

w = [w1-w0 w1 w1+w0]';
A = [2*cos(2*w) 2*cos(w) ones(3,1)];
b = [1/sqrt(2) 1 1/sqrt(2)]';
alpha = inv(A)*b;

h = [alpha ; alpha(end-1:-1:1)].';
[v,ww] = freqz(h,1,1024);
plot(ww/pi,abs(v),[0 0.5],[1/sqrt(2) 1/sqrt(2)],'r--',...
    w(1)/pi,1/sqrt(2),'rp',w(3)/pi,1/sqrt(2),'rp',w(2)/pi,1,'rp');
grid on
xlabel('normalized frequency (times \pi)');
ylabel('magnitude');
axis([0 1 0 10]);
```

This is not a very good bandpass filter. Even though it meets the specifications, this looks more like a highpass filter.

(b) 2 points. Discuss what happens as ω_0 gets small.
Solution: The following plot shows what happens when $\omega_0 = \pi/8, \pi/12, \pi/16, \pi/32$, including the magnitude response and the z-plane.





The magnitude response shows that the "passband" of our bandpass filter is becoming narrower, as we expected, but that the filter is becoming more like a bandstop filter as ω_0 gets small since the low and high frequencies are passed with higher gain than one. The z-plane plots are also interesting. We see the zeros moving closer to $e^{\pm j\omega_1}$ as ω_0 gets small, but in order to keep the magnitude response of the filter unity at ω_1 the overall gain of the filter must increase. This explains why the high and low frequency gains of the filter increase as ω_0 gets small. Overall, this isn't a very good way to design a narrowband bandpass filter.

4. 3 points. Suppose $x[n]$ is a length-N sequence and $y[n]$ is a length-$2N$ sequence formed by repeating $x[n]$ twice, i.e.

$$y[n] = \begin{cases} x[n] & n = 0, 1, \dots, N-1 \\ x[n-N] & n = N, \dots, 2N-1. \end{cases}$$

Let $Y[k]$ be the $2N$-point DFT of $y[n]$ for $k = 0, \dots, 2N-1$ and let $Z[k] = Y[2k]$ for $k = 0, 1, \dots, N-1$. Relate $z[n] = \mathsf{IDFT}\{Z[k]\}$ to $x[n]$ for $n = 0, \dots, N-1$.

Solution: Using what we covered in lecture about periodic extensions, we can say

$$Y[k] = \begin{cases} 2X[k/2] & k = 0, 2, 4, \dots, 2N-2 \\ 0 & k = 1, 3, \dots, 2N-1. \end{cases}$$

Now, since $Z[k] = Y[2k]$ (downsampling in the frequency domain), then $Z[k] = 2X[k]$ for $k = 0, 1, \dots, N-1$, hence $z[n] = 2x[n]$ for $n = 0, 1, \dots, N-1$.

5. 4 points. Mitra 4.43. Suggested approach: Use z-domain analysis to find the zero-state response. To find the zero-input response, find the homogeneous solution and apply the initial conditions to solve for the unknown constants. Compute the total response as the sum of the zero-state response and the zero-input response. You can check your answer with Matlab.
Solution: Let's work on the zero-state response first. Given $x[n] = 3^n \mu n$, the transfer function is easily computed as

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1 - 0.7z^{-1} - 0.02z^{-2}}$$

with ROC extending outward from the largest magnitude pole at $z = 0.7275$ and $X(z) = \frac{1}{1-3z^{-1}}$ with ROC $|z| > 3$. Hence

$$Y(z) = \frac{a}{1 - 0.7275z^{-1}} + \frac{b}{1 + 0.0275z^{-1}} + \frac{c}{1 - 3^{-1}}$$

and we need to calculate a, b, and c. I get $a = -0.3085$, $b = 0.0003$, and $c = 1.3081$ using Matlab's `residuez` command. Hence, the zero-state response is

$$y_{zs}[n] = a(0.7275)^n \mu[n] + b(-0.0275)^n \mu[n] + c(3)^n \mu[n].$$

The zero-input response can be determined by writing the complementary solution

$$y_c[n] = \alpha_1 (0.7275)^n + \alpha_2 (-0.0275)^n$$

and solving for α_1 and α_2 based on the given initial conditions $y[-1] = 3$ and $y[-2] = 0$. So we have two simultaneous equations

$$\begin{aligned} 3 &= \alpha_1 (0.7275)^{-1} + \alpha_2 (-0.0275)^{-1} \\ 0 &= \alpha_1 (0.7275)^{-2} + \alpha_2 (-0.0275)^{-2} \end{aligned}$$

and I get $\alpha_1 = 2.1030$ and $\alpha_2 = -0.0030$. Hence, the zero-input response is

$$y_{zi}[n] = \alpha_1 (0.7275)^n + \alpha_2 (-0.0275)^n$$

for $n \geq 0$. Total response is then

$$\begin{aligned} y[n] &= y_{zs}[n] + y_{zi}[n] \\ &= \alpha_1 (0.7275)^n \mu[n] + \alpha_2 (-0.0275)^n \mu[n] + a(0.7275)^n \mu[n] + b(-0.0275)^n \mu[n] + c(3)^n \mu[n] \\ &= 1.7945(0.7275)^n \mu[n] - 0.0027(-0.0275)^n \mu[n] + 1.3081(3)^n \mu[n]. \end{aligned}$$

This can be confirmed in Matlab.

6. 3 points. Compute the impulse response of the system in Mitra 4.43.
Solution: We can use basically the same approach as before except $x[n] = \delta[n]$ and the initial conditions are all zero since the impulse response implies that the system is relaxed. Since there is no zero-input response to worry about, we just have to find the inverse z-transform of the transfer function. This can be done via partial fraction expansion

$$H(z) = \frac{a}{1 - 0.7275z^{-1}} + \frac{b}{1 + 0.0275z^{-1}} = \frac{1}{1 - 0.7z^{-1} - 0.02z^{-2}}.$$

Solving for a and b, I get $a = 0.9636$ and $b = 0.0364$. Hence the impulse response of this system is

$$h[n] = 0.9636(0.7275)^n \mu[n] + 0.0364(-0.0275)^n \mu[n].$$

which can also be confirmed in Matlab.

7. 4 points. Mitra 5.9.

(a) $$Y_a[k] = \sum_{n=0}^{N-1} \alpha^n W_N^{kn} = \sum_{n=0}^{N-1} (\alpha W_N^k)^n = \frac{1-\alpha^N W_N^{kN}}{1-\alpha W_N^k} = \frac{1-\alpha^N}{1-\alpha W_N^k}.$$

(b) Note that $y_b[n] = \begin{cases} +4, & \text{for } n \text{ even}, \\ -2, & \text{for } n \text{ odd}, \end{cases} = 3(-1)^n + 1.$

Hence we can use the result from Part (a) and write:

$$Y_b[k] = \sum_{n=0}^{N-1} [3(-1)^n + 1] W_N^{kn} = 3\sum_{n=0}^{N-1} \left(-W_N^k\right)^n + \sum_{n=0}^{N-1} \left(W_N^k\right)^n.$$

Assume $W_N^k \neq \pm 1$. Then:

$$\begin{aligned} Y_b[k] &= 3\frac{1-\left(-W_N^k\right)^N}{1-\left(-W_N^k\right)} + \frac{1-\left(W_N^k\right)^N}{1-\left(W_N^k\right)} = 3\frac{\left(1-\left(-W_N^k\right)^N\right)\left(1-W_N^k\right)}{\left(1-W_N^k\right)\left(1+W_N^k\right)} + \frac{\left(1-\left(W_N^k\right)^N\right)\left(1+W_N^k\right)}{\left(1-W_N^k\right)\left(1+W_N^k\right)} \\ &= 3\frac{\left(1-(-1)^N e^{j2\pi k}\right)\left(1-W_N^k\right)}{\left(1-W_N^k\right)\left(1+W_N^k\right)} + \frac{\left(1-e^{j2\pi k}\right)\left(1+W_N^k\right)}{\left(1-W_N^k\right)\left(1+W_N^k\right)} \\ &= \frac{\left(3-3W_N^k-3(-1)^N e^{j2\pi k}+3(-1)^N e^{j2\pi k}W_N^k\right)+\left(1+W_N^k-e^{j2\pi k}-e^{j2\pi k}W_N^k\right)}{\left(1-W_N^k\right)\left(1+W_N^k\right)} \\ &= \frac{4-2W_N^k-3(-1)^N e^{j2\pi k}+3(-1)^N e^{j2\pi k}W_N^k-e^{j2\pi k}-e^{j2\pi k}W_N^k}{\left(1-W_N^k\right)\left(1+W_N^k\right)} \\ &= \frac{4-3(-1)^N e^{j2\pi k}-e^{j2\pi k}-W_N^k\left(2-3(-1)^N e^{j2\pi k}+e^{j2\pi k}\right)}{\left(1-W_N^k\right)\left(1+W_N^k\right)}. \end{aligned}$$

Assume $W_N^k = -1 \Leftrightarrow k = N/2$ (where N is necessarily even). Then:

$$Y_b[N/2] = 3\sum_{n=0}^{N-1}(1)^n + \sum_{n=0}^{N-1}(-1)^n = 2N.$$

Now, suppose $W_N^k = 1 \Leftrightarrow k = 0$. Then,

$$Y_b[0] = 3\sum_{n=0}^{N-1}(-1)^n + \sum_{n=0}^{N-1}(1)^n = 3\frac{[1-(-1)^N]}{1-(-1)} + N = \frac{3}{2}[1-(-1)^N] + N = \begin{cases} N, & \text{for } N \text{ even}, \\ 3+N, & \text{for } N \text{ odd}. \end{cases}$$