

## ECE503 Homework Assignment Number 8 Solution

1. 3 points. Recall that an analog integrator has transfer function  $H_a(s) = \frac{1}{s}$ . Use the bilinear transform to find the digital transfer function  $G(z)$  from the parent analog transfer function  $H_a(s)$ . Compare the frequency responses  $H_a(\Omega)$  and  $G(\omega)$  and the impulse responses  $h_a(t)$  and  $g[n]$ .

**Solution:** We simply substitute  $s = \frac{2}{T} \left( \frac{1-z^{-1}}{1+z^{-1}} \right)$  into  $H_a(s)$  to get

$$H(z) = \frac{T}{2} \left( \frac{1+z^{-1}}{1-z^{-1}} \right).$$

Note this is different than the typical digital integrator  $H(z) = \frac{1}{1-z^{-1}}$ . The following Matlab code was used to compare the frequency responses  $H_a(\Omega)$  and  $G(\omega)$  and the impulse responses  $h_a(t)$  and  $g[n]$ .

```
% sampling parameters
T = 1;
Fs = 1/T;

% CT system
% H(s) = 1/s
tfinal = 20;
ctnum = 1;
ctden = [1 0];
ctsys = tf(ctnum,ctden);
[y,t] = impulse(ctsys,tfinal);
Omega = linspace(0,pi*Fs,1024);
h = freqs(ctnum,ctden,Omega);

% DT system via bilinear transform
dtnum = [T/2 T/2];
dtden = [1 -1];
dtsys = tf(dtnum,dtden,T);
[y2,t2] = impulse(dtsys,tfinal);
h2 = freqz(dtnum,dtden,Omega/Fs);

% plots
subplot(1,2,1);
plot(t,y,'r')
hold on
stem(t2,y2);
hold off
```

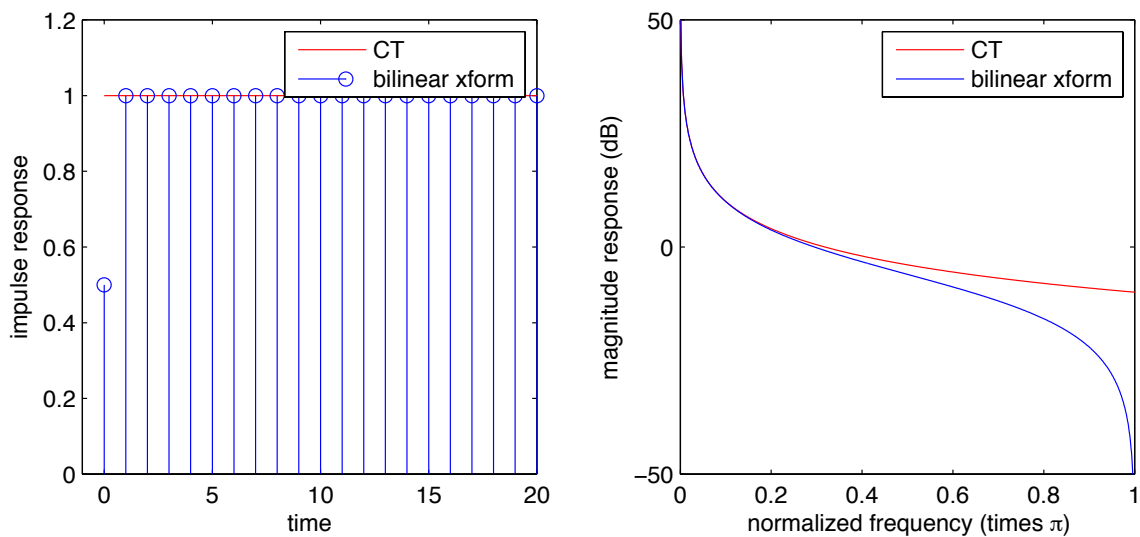
```

xlabel('time');
ylabel('impulse response');
legend('CT','bilinear xform');
axis([-1 tfinal 0 1.2]);
axis square

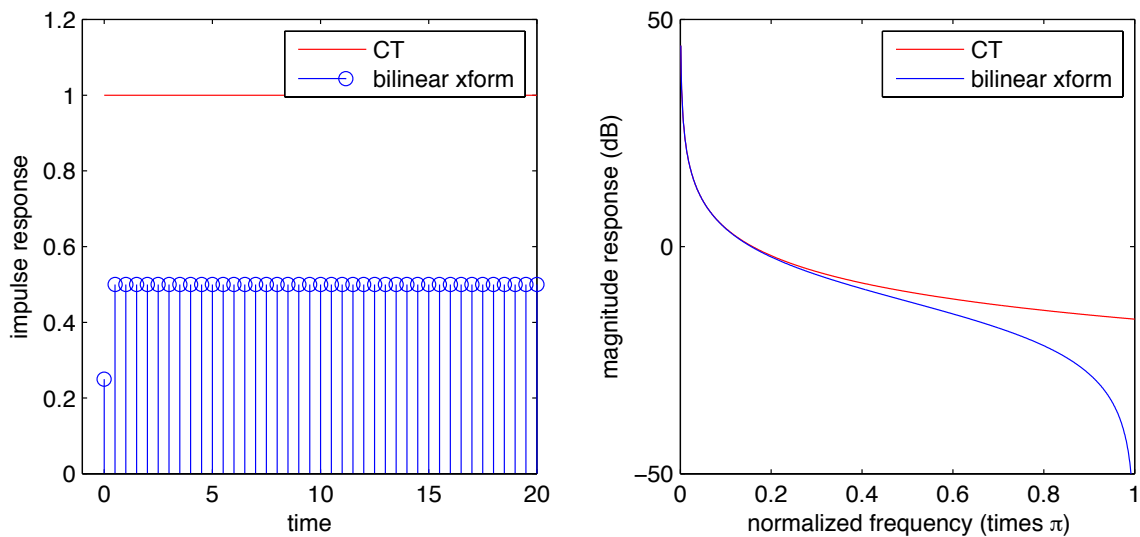
subplot(1,2,2);
plot(Omega/Fs/pi,20*log10(abs(h)), 'r',Omega/Fs/pi,20*log10(abs(h2)), 'b')
xlabel('normalized frequency (times \pi)');
ylabel('magnitude response (dB)');
legend('CT','bilinear xform');
axis([0 1 -50 50]);
axis square

```

The impulse and frequency responses for  $T = 1$  are shown below.



If we change  $T$  to be  $T = 0.5$ , we get the following impulse and frequency responses.



Note the scaling of the impulse response. Also note the magnitude response is changed for both the CT and bilinear transformed DT systems (the CT system's magnitude response has changed because the normalized frequency range now corresponds to a larger range of non-normalized CT frequencies). When  $T = 0.5$ , the DT system accurately approximates the CT system over a larger range of non-normalized CT frequencies.

2. 3 points. Mitra 9.8.

**Solution:**

$$H_a(s) = \frac{A}{s + \alpha}. \text{ The transfer function has a pole at } s = -\alpha. \text{ Now}$$

$$G(z) = \text{Residue at } s = -\alpha \left[ \frac{A}{(s + \alpha)(1 - e^{sT} z^{-1})} \right] = \frac{A}{1 - e^{sT} z^{-1}} \Big|_{s = -\alpha} = \frac{A}{1 - e^{-\alpha T} z^{-1}}.$$

3. 3 points. Mitra 9.15.

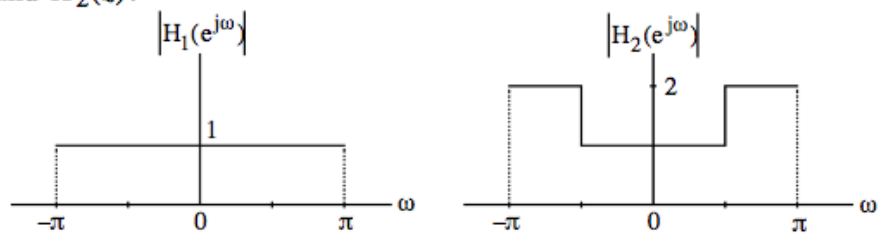
**Solution:**

The mapping is given by  $s = \frac{1}{T}(1 - z^{-1})$  or equivalently, by  $z = \frac{1}{1 - sT}$ . For  $s = \sigma_o + j\Omega_o$ ,  
 $z = \frac{1}{1 - \sigma_o T - j\Omega_o T}$ . Therefore,  $|z|^2 = \frac{1}{(1 - \sigma_o T)^2 + (\Omega_o T)^2}$ . Hence,  $|z| < 1$  for  $\sigma_o < 0$ . As  
a result, a stable  $H_a(s)$  results in a stable  $H(z)$  after the transformation. However, for  
 $\sigma_o = 0$ ,  $|z|^2 = \frac{1}{1 + (\Omega_o T)^2}$  which is equal to 1 only for  $\Omega_o = 0$ . Hence, only the point  $\Omega_o = 0$   
on the  $j\Omega$ -axis in the  $s$ -plane is mapped onto the point  $z = 1$  on the unit circle.  
Consequently, this mapping is not useful for the design of digital filters via analog filter  
transformation.

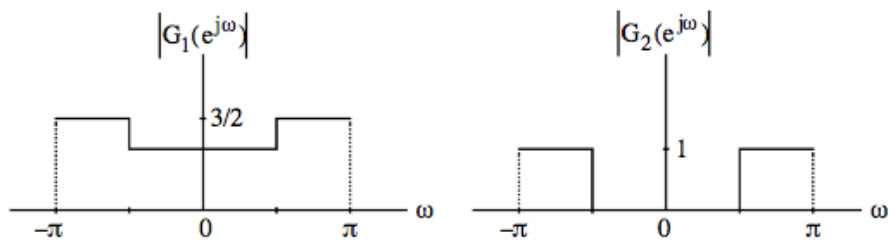
4. 3 points. Mitra 9.16

Solution:

For no aliasing  $T \leq \frac{\pi}{\Omega_c}$ . Figure below shows the magnitude responses of the digital filters  $H_1(z)$  and  $H_2(z)$ .



(a) The magnitude responses of the digital filters  $G_1(z)$  and  $G_2(z)$  are shown below:



(b) As can be seen from the above  $G_1(z)$  is a multi-passband filter, whereas,  $G_2(z)$  is a highpass filter.

5. 5 points. Mitra 9.18

Solution:

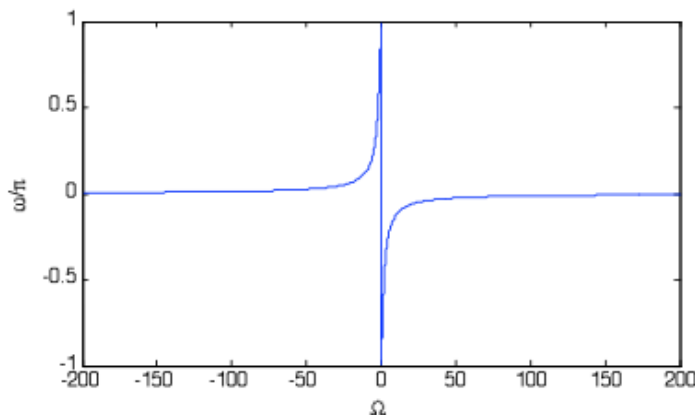
$$(a) \quad s = \frac{2}{T} \left( \frac{1+z^{-1}}{1-z^{-1}} \right) \Rightarrow z^{-1} = \frac{s-2/T}{s+2/T}, \text{ or } z = \frac{s+2/T}{s-2/T} = \frac{(\sigma+2/T) + j\Omega}{(\sigma-2/T) + j\Omega}.$$

$$(b) \quad |z|^2 = \frac{(\sigma+2/T)^2 + \Omega^2}{(\sigma-2/T)^2 + \Omega^2}, \text{ so } |z|^2 \Big|_{\sigma+j\Omega, \sigma=0} = 1 \text{ and } |z|^2 \Big|_{s=\sigma+j\Omega, \sigma<0} < 1. \text{ The}$$

first proves that a point on the  $j\Omega$  axis is mapped to a point on the unit circle, and the second proves that a point in the left-half  $s$ -plane is mapped to a point inside the unit circle (stability is preserved). This mapping does indeed have all the desirable properties.

(c) If  $s(f_1(z))$  is the bilinear transformation in Eq. (9.14) and  $s(f_2(z))$  is the bilinear transformation in Eq. (9.72), then  $s(f_2(z)) = s(-f_1(z))$ .

$$(d) \quad j\Omega = \frac{2}{T} \frac{1+e^{-j\omega}}{1-e^{-j\omega}} \Rightarrow \omega = -2 \cot^{-1} \left( \frac{\Omega T}{2} \right).$$



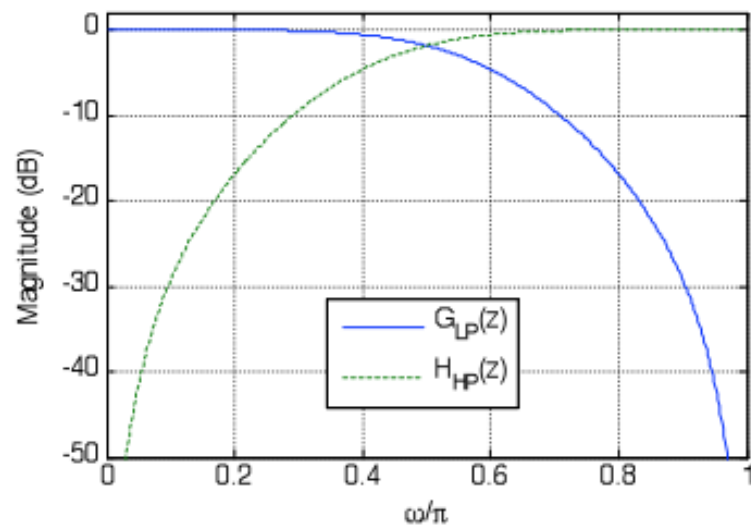
(e)  $G(z)$  is a high-pass filter because small  $\Omega$  (large  $\omega$ ) frequencies passed while large  $\Omega$  (small  $\omega$ ) frequencies are attenuated.

6. 3 points. Mitra 9.28

Solution:

$$\alpha = \frac{-\cos\left(\frac{0.55\pi+0.45\pi}{2}\right)}{\cos\left(\frac{0.55\pi-0.45\pi}{2}\right)} = 0.$$

$$H_{HP}(z) = G_{LP}(z)\Big|_{z^{-1} = -\left(\frac{\hat{z}^{-1} + \alpha}{1 + \alpha\hat{z}^{-1}}\right)} = \frac{0.3404 - 0.6808\hat{z}^{-1} + 0.3404\hat{z}^{-2}}{1 - 0.1842\hat{z}^{-1} + 0.1776\hat{z}^{-2}}.$$



7. 5 points. Mitra M9.11.

**Solution to part (a):**

```
% Mitra M9.11
Fs = 12e3;

% compute normalized frequencies
f = [0.9 1.5 2.5 4]*1e3;
omega = f/Fs*2*pi;
T = 1; % arbitrary
Omega_prewarped = (2/T)*tan(omega/2);

% convert CT frequency specs to a prototype CT lowpass filter
% we want the two stopband frequencies to be GEOMETRICALLY symmetric
% around the GEOMETRIC center frequency of the passband
Omega_0 = sqrt(Omega_prewarped(2)*Omega_prewarped(3));
BW = Omega_prewarped(3)-Omega_prewarped(2);

% adjust upper stopband edge so that product of stopband edge frequencies
% matches Omega_0^2
Omega_prewarped(4) = Omega_0^2/Omega_prewarped(1);
```

This results in the CT BPF edge frequencies (using  $T = 1$ , your answer will be different for different choices of  $T$ ):

```
Omega_prewarped =
    0.4802    0.8284    1.5347    2.6478
```

with maximum passband ripple of 0.7 dB and minimum stopband attenuation of 42 dB.

**Solution to part (b):** Continuing from my previous code, I used this code to determine the parameters for the prototype CT LPF:

```
% create prototype CT LPF filter specs (see Appendix B)
Omega_p = 1;
Omega_s = (Omega_0^2-Omega_prewarped(1)^2)/(Omega_prewarped(1)*BW);
```

This gives  $\Omega_p = 1$ ,  $\Omega_s = 3.0693$ , with maximum passband ripple of 0.7 dB and minimum stopband attenuation of 42 dB.

**Solution to part (c):** Continuing from my previous code, I used this code to compute the prototype Type 1 Chebychev lowpass filter, then transform that LPF to a BPF, then finally transform the CT BPF to a DT BPF:

```
% design prototype CT LPF
[N, Wn] = cheb1ord(Omega_p, Omega_s, 0.7, 42, 's');
[B, A] = cheby1(N, 0.7, Wn, 's');

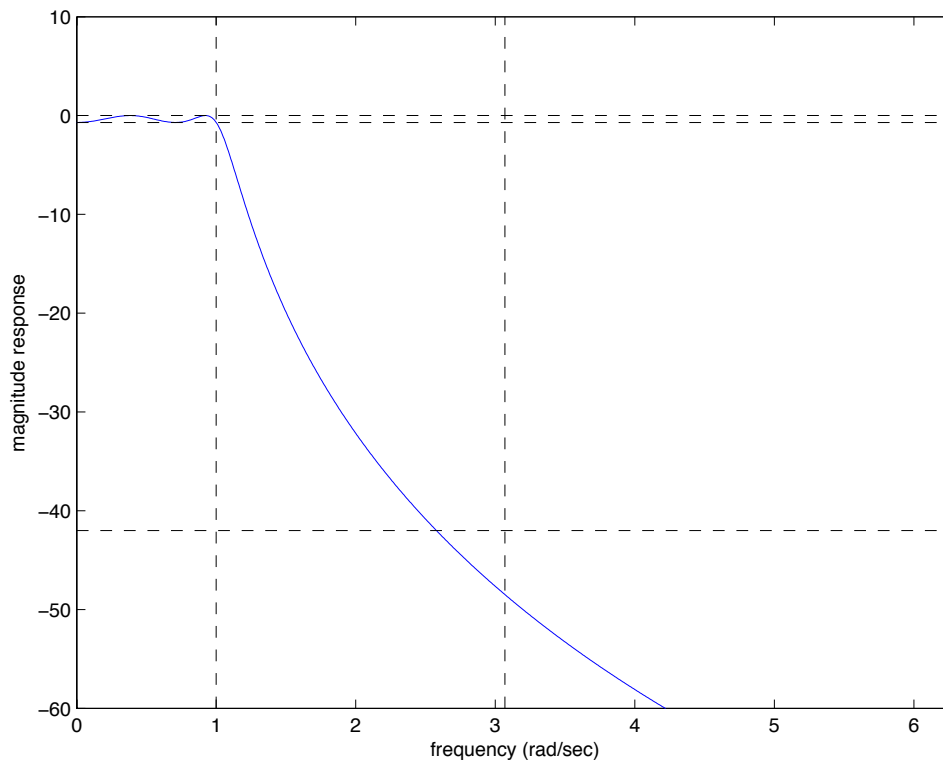
% convert to CT BPF
[BT,AT] = lp2bp(B,A,Omega_0,BW);
```

```
% convert to DT BPF
[num,den] = bilinear(BT,AT,T);
```

Here is my code to plot the magnitude response of the prototype CT LPF.

```
% look at frequency response of prototype CT LPF
w = linspace(0,2*pi,1024);
h = freqs(B,A,w);
plot(w,20*log10(abs(h)));
hold on
plot([Omega_p Omega_p],[-60 10],'k--',...
     [Omega_s Omega_s],[-60 10],'k--',...
     [0 2*pi],[0 0],'k--',...
     [0 2*pi],[-0.7 -0.7],'k--',...
     [0 2*pi],[-42 -42],'k--');
hold off
axis([0 2*pi -60 10]);
xlabel('frequency (rad/sec)');
ylabel('magnitude response');
```

And here is the plot showing the CT LPF prototype meets the specs.



Here is my code to plot the magnitude response of the CT BPF.

```
% look at frequency response of CT BPF
```

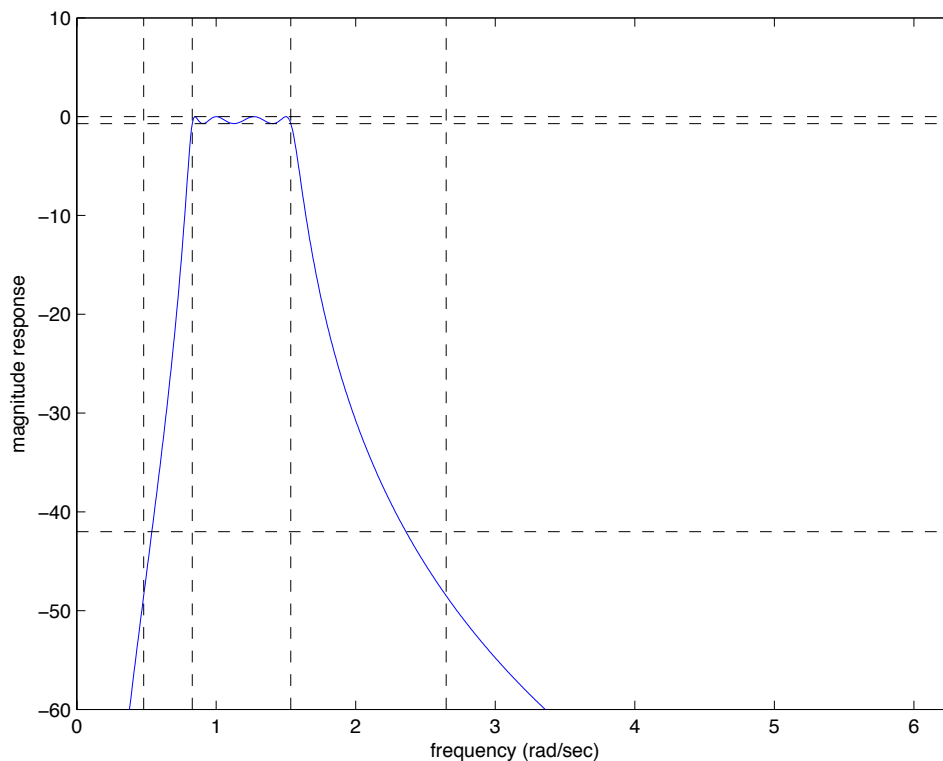


```

w = linspace(0,2*pi,1024);
h = freqs(BT,AT,w);
plot(w,20*log10(abs(h)));
hold on
plot([Omega_prewarped(1) Omega_prewarped(1)],[-60 10], 'k--',...
      [Omega_prewarped(2) Omega_prewarped(2)],[-60 10], 'k--',...
      [Omega_prewarped(3) Omega_prewarped(3)],[-60 10], 'k--',...
      [Omega_prewarped(4) Omega_prewarped(4)],[-60 10], 'k--',...
      [0 2*pi],[0 0], 'k--',...
      [0 2*pi],[-0.7 -0.7], 'k--',...
      [0 2*pi],[-42 -42], 'k--');
hold off
axis([0 2*pi -60 10]);
xlabel('frequency (rad/sec)');
ylabel('magnitude response');

```

And here is the plot.



Here is my code to plot the magnitude response of the final DT filter:

```

% plot
Fs = 1/T;
w_s2 = 2*atan(Omega_prewarped(4)*T/2);
[h1,f1] = freqz(num,den,1024,Fs);
plot(f1*2,20*log10(abs(h1)))
hold on

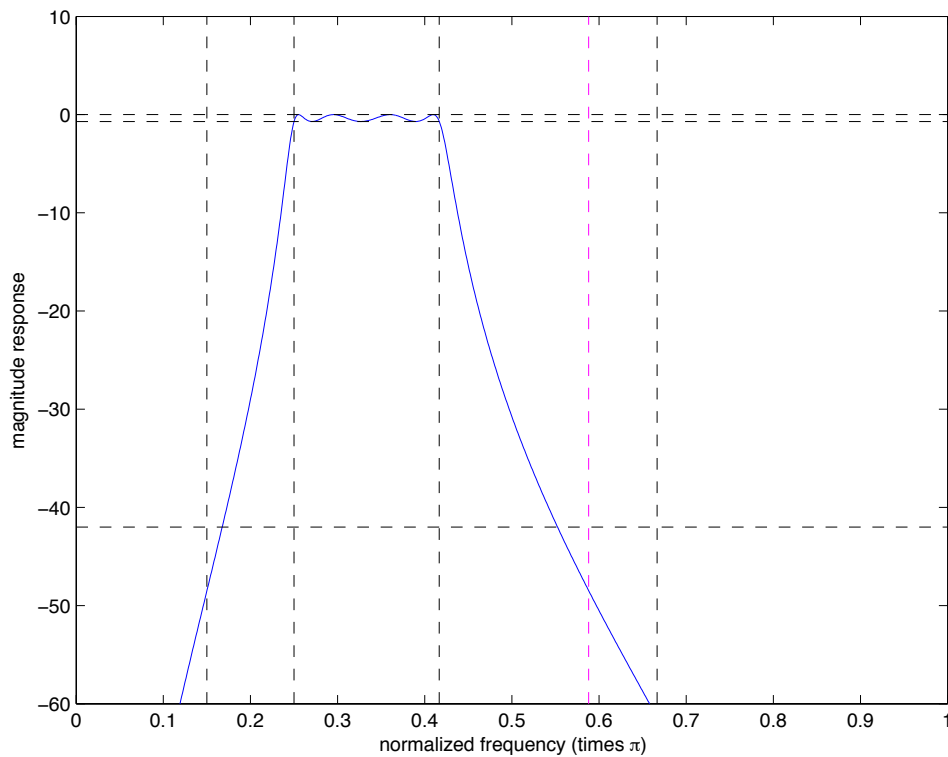
```

```

plot([omega(1) omega(1)]/pi, [-60 10], 'k--', ...
     [omega(2) omega(2)]/pi, [-60 10], 'k--', ...
     [omega(3) omega(3)]/pi, [-60 10], 'k--', ...
     [omega(4) omega(4)]/pi, [-60 10], 'k--', ...
     [w_s2/pi w_s2/pi], [-60 10], 'm--', ...
     [0 1], [0 0], 'k--', ...
     [0 1], [-0.7 -0.7], 'k--', ...
     [0 1], [-42 -42], 'k--');
hold off
axis([0 1 -60 10]);
xlabel('normalized frequency (times \pi)');
ylabel('magnitude response');

```

And here is the plot showing how the final filter meets the specs:



Note the magenta dashed line is the adjusted upper stopband frequency, which we adjusted to be geometrically symmetric with the lower stopband frequency around the geometric center frequency.