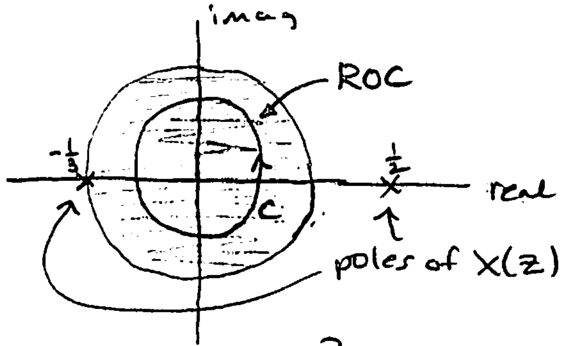
When to use residues?

In lecture 4 I mentioned that it might be convenient to use residues if the number of poles inside the contour C is small since

$$x[n] = \sum_{\lambda_k \text{ inside } C} \mathrm{Res}\left(X(z) z^{n-1}, \lambda_k, m_k\right)$$

But what if $X(z)$ doesn't have any poles inside the contour C?



In this case, does $x[n] = 0$?

Answer: Yes, but we need to account for the poles in $F(z) = X(z) z^{n-1}$, not $X(z)$.

Suppose $X(z) = \dfrac{1}{1+\frac{1}{3}z^{-1}} + \dfrac{1}{1-\frac{1}{2}z^{-1}}$ with ROC: $|z| < \frac{1}{3}$

(same $X(z)$ as shown in lecture examples but different ROC).

If C is a contour $z(\theta) = \frac{1}{4} e^{j\theta}$ $\quad 0 \le \theta \le 2\pi$ it is clear there are no poles of $X(z)$ inside this contour.

But there may be poles of $F(z) = X(z) z^{n-1}$ inside this contour, which is what matters

We can rewrite $X(z) = \dfrac{2z\left(z - \frac{1}{12}\right)}{\left(z+\frac{1}{3}\right)\left(z-\frac{1}{2}\right)}$

For $n = 0, 1, 2, \ldots$ the poles of $F(z)$ are the same as $X(z)$

So $x[n] = 0$ for all $n = 0, 1, \ldots$ (no poles of $F(z)$ inside C)

For $n = -1$, we have $F(z) = \dfrac{2\left(z - \frac{1}{12}\right)}{z\left(z+\frac{1}{3}\right)\left(z-\frac{1}{2}\right)}$

new pole at $z = 0$ (algebraic multiplicity one)

Now $x[-1] = \text{Res}\left(\frac{2(z-\frac{1}{12})}{z(z+\frac{1}{3})(z-\frac{1}{2})}, 0, 1 \right)$

$$= \frac{1}{0!} \left[z F(z) \right]_{z=0}$$

$$= \frac{2(0-\frac{1}{12})}{(0+\frac{1}{3})(0-\frac{1}{2})} = 1.$$

In general, for $n = \ldots, -2, -1$ we have

$$F(z) = \frac{2(z-\frac{1}{12})}{z^{-n}(z+\frac{1}{3})(z-\frac{1}{2})}$$

pole at $z=0$ with algebraic multiplicity $m=-n$.

For $n = \ldots, -2, -1$

$$x[n] = x[-m] = \text{Res}\left(\frac{2(z-\frac{1}{12})}{z^m(z+\frac{1}{3})(z-\frac{1}{2})}, 0, m \right)$$

$$= \frac{1}{(m-1)!} \left[\frac{d^{m-1}}{dz^{m-1}} \left(z^m F(z) \right) \right]_{z=0}$$

$$= \frac{1}{(m-1)!} \left[\frac{d^{m-1}}{dz^{m-1}} \frac{2(z-\frac{1}{12})}{(z+\frac{1}{3})(z-\frac{1}{2})} \right]_{z=0}$$

$$\frac{d}{dz} \frac{f}{g} = \frac{f'g - g'f}{g^2}$$

It might be possible to work out a general expression for the mth derivative, but this looks pretty tedious.

Even in this case, it would probably be simpler to just do a partial fraction expansion and use the ROC + table lookup to write

$$x[n] = -\left(-\tfrac{1}{3}\right)^n \mu[-n-1] - \left(\tfrac{1}{2}\right)^n \mu[-n-1]$$