Digital Signal Processing
Eigenfunctions of DT-LTI Systems

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Linear Algebra: The Concept of an Eigenvector

In linear algebra, we say the vector x is an **eigenvector** of the square matrix A if

$$Ax = \lambda x$$

where λ is a scalar called an eigenvalue.

The vector x is a special vector because, when operated on by A, it is unchanged except for a scale factor.

Example: Suppose

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} \qquad x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad x_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

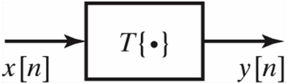
We can calculate

$$Ax_1 = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \text{ and } Ax_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Is x_1 and eigenvector? Is x_2 an eigenvector?

Eigenfunctions of LTI Systems

Can we extend this concept of eigenvectors to **eigenfunctions** of systems?



In other words, are there input sequences $\{x[n]\}$ such that

$$y[n] = T\{x[n]\} = \lambda x[n]$$

where λ is a scalar when T is a DT-LTI system?

Examples

Suppose we have a simple two-sample moving average system

$$y[n] = \frac{x[n] + x[n-1]}{2}.$$

You can easily confirm this system is linear and time invariant.

Suppose $x[n] = \delta[n]$. Is this an eigenfunction?

Suppose $x[n] = u[n]$. Is this an eigenfunction?

Suppose $x[n] = e^{j\omega n}$. Is this an eigenfunction? Note that

$$y[n] = \frac{e^{j\omega n} + e^{j\omega(n-1)}}{2} = \frac{e^{j\omega n} + e^{j\omega n}e^{-j\omega}}{2} = \underbrace{\frac{1 + e^{-j\omega}}{2}}_{\lambda} \cdot \underbrace{e^{j\omega n}}_{x[n]}$$

Eigenfunctions of DT-LTI Systems

In general, for a DT-LTI system with impulse response $h[n]$ and input $x[n] = e^{j\omega n}$, we have

$$
\begin{aligned}
y[n] &= \sum_{k=-\infty}^{\infty} h[k]x[n-k] \\
&= \sum_{k=-\infty}^{\infty} h[k]e^{j\omega(n-k)} \\
&= \sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k}e^{j\omega n} \\
&= e^{j\omega n} \sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k} \\
&= e^{j\omega n} H(e^{j\omega})
\end{aligned}
$$

where $H(e^{j\omega})$ is just a complex number that represents the **frequency response** of the LTI system at the (discrete-time) frequency ω.

Frequency Response

Since $H(e^{j\omega})$ is complex, we can represent it in terms of real and imaginary components

$$H(e^{j\omega}) = H_R(e^{j\omega}) + jH_I(e^{j\omega})$$

or (more commonly) in terms of its magnitude and phase

$$H(e^{j\omega}) = |H(e^{j\omega})|e^{j\angle H(e^{j\omega})}.$$

Using the latter representation, note that

$$\begin{aligned} y[n] &= e^{j\omega n}|H(e^{j\omega})|e^{j\angle H(e^{j\omega})} \\ &= |H(e^{j\omega})|e^{j(\omega n + \angle H(e^{j\omega}))}. \end{aligned}$$

The output of the system is a complex exponential at the same frequency as the input. Only the magnitude and phase are changed.