Digital Signal Processing
Examples and Applications of the Frequency Response

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Response of DT-LTI Systems to Sinusoidal Inputs (1 of 2)

Suppose we have an input sequence $x[n] = A\cos(\omega n + \phi)$ for all $n \in \mathbb{Z}$. We can use Euler's identity to write

$$\begin{aligned} A\cos(\omega n + \phi) &= \frac{A}{2}\left(e^{j(\omega n+\phi)} + e^{-j(\omega n+\phi)}\right) \\ &= \frac{A}{2}e^{j\phi}e^{j\omega n} + \frac{A}{2}e^{-j\phi}e^{-j\omega n} \end{aligned}$$

Passing this signal through an DT-LTI system with impulse response $h[n]$ results in

$$\begin{aligned} y[n] &= \frac{A}{2}e^{j\phi}H(e^{j\omega})e^{j\omega n} + \frac{A}{2}e^{-j\phi}H(e^{-j\omega})e^{-j\omega n} \\ &= \frac{A}{2}e^{j\phi+\angle H(e^{j\omega})}|H(e^{j\omega})|e^{j\omega n} + \frac{A}{2}e^{-j\phi+\angle H(e^{-j\omega})}|H(e^{-j\omega})|e^{-j\omega n} \end{aligned}$$

We can simplify this a bit with an additional assumption...

Response of DT-LTI Systems to Sinusoidal Inputs (2 of 2)

Let's assume the impulse response $h[n]$ is real-valued. This implies $|H(e^{-j\omega})| = |H(e^{j\omega})|$ and $\angle H(e^{-j\omega}) = -\angle H(e^{j\omega})$.

Then

$$\begin{aligned} y[n] &= \frac{A}{2} e^{j\phi + \angle H(e^{j\omega})} |H(e^{j\omega})| e^{j\omega n} + \frac{A}{2} e^{-j\phi + \angle H(e^{-j\omega})} |H(e^{-j\omega})| e^{-j\omega n} \\ &= |H(e^{j\omega})| \frac{A}{2} \left(e^{j\phi + \angle H(e^{j\omega})} e^{j\omega n} + e^{-j\phi - \angle H(e^{j\omega})} e^{-j\omega n} \right) \\ &= |H(e^{j\omega})| A \cos\left(\omega n + \phi + \angle H(e^{j\omega}) \right) \end{aligned}$$

Hence, given an input sequence $x[n] = A\cos(\omega n + \phi)$ for all $n \in \mathbb{Z}$, the output sequence is the same sinusoidal sequence with two differences:

- Amplitude scaled by $|H(e^{j\omega})|$
- Phase shifted by $\angle H(e^{j\omega})$

The frequency of the output is identical to the frequency of the input. No new frequencies are generated.

Simple Filtering (1 of 2)

Problem: We have an input signal

$$x[n] = c_0 \cos(\omega_0 n + \phi_0) + c_1 \cos(\omega_1 n + \phi_1).$$

We want to design an DT-LTI system that blocks the signal at ω_0 and passes the signal at ω_1.

Approach: Assume a real-valued symmetric impulse response

$$h[n] = \{\alpha_0, \alpha_1, \alpha_0\}.$$

We want to find values for α_0 and α_1 so that $|H(e^{j\omega_0})| = 0$ and $|H(e^{j\omega_1})| = 1$. Two equations and two unknowns.

Simple Filtering (2 of 2)

To compute the values of α_0 and α_1 that achieve the desired goal, we first compute the frequency response

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n} = \alpha_0 + \alpha_1 e^{-j\omega} + \alpha_0 e^{-j2\omega} = (2\alpha_0 \cos(\omega) + \alpha_1)e^{-j\omega}$$

Note $|H(e^{j\omega})| = |2\alpha_0 \cos(\omega) + \alpha_1|$. Hence, we can achieve the desired goal of $|H(e^{j\omega_0})| = 0$ and $|H(e^{j\omega_1})| = 1$ if

$$\begin{aligned} 2\alpha_0 \cos(\omega_0) + \alpha_1 &= 0 \\ 2\alpha_0 \cos(\omega_1) + \alpha_1 &= 1. \end{aligned}$$

These simultaneous equations are not difficult to solve for α_0:

$$2(\cos(\omega_1) - \cos(\omega_0))\alpha_0 = 1 \qquad \Leftrightarrow \qquad \alpha_0 = \frac{1}{2(\cos(\omega_1) - \cos(\omega_0))}$$

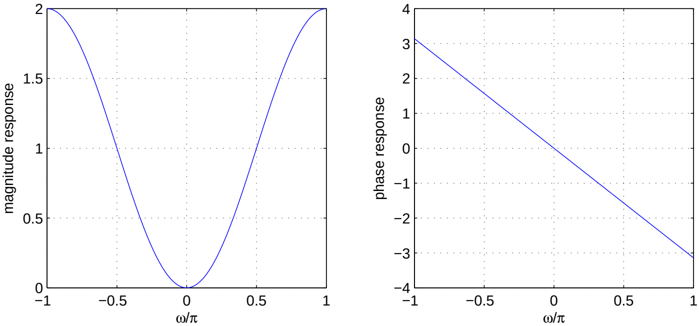
Then plug this result back into one of the equations above to get α_1.

Simple Filtering Numerical Example

Suppose $\omega_0 = 0$ and $\omega_1 = \frac{\pi}{2}$. We need to solve

$$2\alpha_0 \cos(0) + \alpha_1 = 0$$
$$2\alpha_0 \cos(\pi/2) + \alpha_1 = 1$$

which results in $h[n] = \{\alpha_0, \alpha_1, \alpha_0\} = \{-1/2, 1, -1/2\}$. Using Matlab's `freqz` function, we can plot:



Linear Phase Systems

Definition

A **linear phase system** is a system with phase response $\theta(\omega) = \angle H(e^{j\omega}) = -c\omega$ for all ω and any constant c.

For example, suppose we have an DT-LTI system with impulse response

$$h[n] = \{1, 2, 1\}.$$

We can compute the frequency response

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n} = 1 + 2e^{-j\omega} + 1e^{-j2\omega} = (2\cos(\omega) + 2)e^{-j\omega}$$

We see that $\theta(\omega) = \angle H(e^{j\omega}) = -\omega$. Is this a linear phase system?

Linear phase systems are important because all frequencies are delayed by the same amount of time.