Digital Signal Processing
Frequency Response of DT-LTI Systems

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Computing $H(e^{j\omega})$ from a Linear Constant-Coefficient Difference Equation

Suppose we have a system defined by

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{m=0}^{M} b_m x[n-m]$$

for finite N and M. How can we determine $H(e^{j\omega})$?

One approach: First determine the impulse response $h[n]$ and then compute

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h[k] e^{-j\omega k}$$

Computing $H(e^{j\omega})$ from a Linear Constant-Coefficient Difference Equation

We can also solve this more directly. Recall that, when $x[n] = e^{j\omega n}$, we have $y[n] = H(e^{j\omega})e^{j\omega n}$. By simple time-shifting, we can write

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{m=0}^{M} b_m x[n-m]$$

$$\sum_{k=0}^{N} a_k H(e^{j\omega}) e^{-jk\omega} e^{j\omega n} = \sum_{m=0}^{M} b_m e^{-jm\omega} e^{j\omega n}$$

We can cancel the common $e^{j\omega n}$ terms and solve for $H(e^{j\omega})$ get the direct result

$$H(e^{j\omega}) = \frac{\sum_{m=0}^{M} b_m e^{-jm\omega}}{\sum_{k=0}^{N} a_k e^{-jk\omega}}$$

You can also use this result to go from $H(e^{j\omega})$ to a linear constant-coefficient difference equation.

Computation of the Frequency Response in Matlab

The `freqz` function in Matlab is handy for computing $H(e^{j\omega})$ at various values of $\omega \in [-\pi, \pi)$. You just pass in vectors `b`, `a`, and `w`. For example:

```
b = [-0.5, 1, -0.5];               % b coefficients
a = [1, 0.5, 0.25];                % a coefficients
omega = linspace(-pi,pi,1024);     % omega values
[h,w] = freqz(b,a,omega);          % compute H(e^jomega)
subplot(1,2,1);
plot(w/pi,abs(h));                 % magnitude response
xlabel('\omega/\pi');
ylabel('magnitude response');
grid on; axis square;
subplot(1,2,2);
plot(w/pi,unwrap(angle(h)));       % phase response
xlabel('\omega/\pi');
ylabel('phase response');
grid on; axis square;
```

Computing $H(e^{j\omega})$ from $h[n]$ (part 1 of 2)

Suppose now we only have the impulse response $h[n]$ and no input/output description of the DT-LTI system. We know

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k} \tag{1}$$

but it is not clear that this sum always converges. Under what conditions is $|H(e^{j\omega})| < \infty$ for all ω?

Condition 1: If the system has finite impulse response, i.e., $h[k] = 0$ except on a finite number of time indices, then (1) is clearly finite for all ω.

Condition 2: If the impulse response is absolutely summable, i.e., $\sum_k |h[k]| = B_h < \infty$, then (1) is finite for all ω. This follows from

$$|H(e^{j\omega})| = \left|\sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k}\right| \le \sum_{k=-\infty}^{\infty} |h[k]||e^{-j\omega k}| = B_h < \infty$$

Computing $H(e^{j\omega})$ from $h[n]$ (part 2 of 2)

The prior conditions are both sufficient but not necessary for the existence of $H(e^{j\omega})$. If we allow for Dirac delta functions in $H(e^{j\omega})$, we can express the frequency response of an even wider class of $h[n]$.

It is useful to have the inverse frequency response

$$h[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} H(e^{j\omega}) e^{j\omega n}\, d\omega$$

and work backwards. For example, suppose $H(e^{j\omega}) = 2\pi\delta(\omega)$. Then

$$h[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} 2\pi\delta(\omega) e^{j\omega n}\, d\omega = 1$$

Recall however that $H(e^{j\omega})$ must be periodic in ω with period 2π. So $H(e^{j\omega}) = 2\pi\delta(\omega)$ is not valid. This is easy to fix by writing

$$H(e^{j\omega}) = 2\pi \sum_{r=-\infty}^{\infty} \delta(\omega + 2\pi r)$$

which is valid and is in fact the frequency response of $h[n] = 1$ for all n.

Summary: The DTFT

Since the impulse response is just a DT sequence, we can define the discrete-time Fourier transform (DTFT) of any DT sequence $x[n]$ as

$$X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} x[k]e^{-j\omega k}$$

and the inverse DTFT as

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}\,d\omega$$

Remarks:

- $x[n]$ is defined for all $n \in \mathbb{Z}$.
- $X(e^{j\omega})$ is defined for all $\omega \in \mathbb{R}$ but we usually only plot $-\pi < \omega \leq \pi$.
- $X(e^{j(\omega+r2\pi)}) = X(e^{j\omega})$ for all $r \in \mathbb{Z}$.
- $X(e^{j\omega})$ is complex (except in special cases), even if $x[n]$ is real.