Digital Signal Processing
Discrete-Time Fourier Transform Examples

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DTFT Example 1 (part 1 of 3)

Suppose

$$x[n] = a^n \cos(\omega_0 n + \phi)u[n]$$

with $|a| < 1$ and $-\pi < \omega_0 \leq \pi$. Find $X(e^{j\omega})$.

From our DTFT table, we have

$$w[n] = a^n u[n] \leftrightarrow \frac{1}{1 - ae^{-j\omega}}$$

and

$$v[n] = \cos(\omega_0 n + \phi) \leftrightarrow \sum_{k=-\infty}^{\infty} \left[\pi e^{j\phi}\delta(\omega - \omega_0 + k2\pi) + \pi e^{-j\phi}\delta(\omega + \omega_0 + k2\pi)\right]$$

We can use the modulation/windowing property to compute

$$X(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} V(e^{jt})W(e^{j(\omega-t)})\,dt$$

continued...

DTFT Example 1 (part 2 of 3)

$$\begin{aligned}
X(e^{j\omega}) &= \frac{1}{2}\int_{-\pi}^{\pi}\left(\sum_{k=-\infty}^{\infty}\left[e^{j\phi}\delta(t-\omega_0+k2\pi)+e^{-j\phi}\delta(t+\omega_0+k2\pi)\right]\right)\frac{1}{1-ae^{-j(\omega-t)}}\,dt \\
&= \frac{1}{2}\int_{-\pi}^{\pi}\left[e^{j\phi}\delta(t-\omega_0)+e^{-j\phi}\delta(t+\omega_0)\right]\frac{1}{1-ae^{-j(\omega-t)}}\,dt \\
&= \frac{e^{j\phi}}{2}\int_{-\pi}^{\pi}\delta(t-\omega_0)\frac{1}{1-ae^{-j(\omega-t)}}\,dt+\frac{e^{-j\phi}}{2}\int_{-\pi}^{\pi}\delta(t+\omega_0)\frac{1}{1-ae^{-j(\omega-t)}}\,dt \\
&= \frac{e^{j\phi}}{2}\frac{1}{1-ae^{-j(\omega-\omega_0)}}+\frac{e^{-j\phi}}{2}\frac{1}{1-ae^{-j(\omega+\omega_0)}}
\end{aligned}$$

DTFT Example 1 (part 3 of 3)

Alternative approach: Use Euler's identity to write

$$\begin{aligned} x[n] &= a^n \cos(\omega_0 n + \phi) u[n] \\ &= a^n \left(\frac{e^{j(\omega_0 n + \phi)} + e^{-j(\omega_0 n + \phi)}}{2} \right) u[n] \\ &= \frac{1}{2} e^{j\phi} (a e^{j\omega_0})^n u[n] + \frac{1}{2} e^{-j\phi} (a e^{-j\omega_0})^n u[n] \end{aligned}$$

Using linearity and the DTFT table result we used earlier, we can immediately write

$$\begin{aligned} X(e^{j\omega}) &= \frac{1}{2} e^{j\phi} \frac{1}{1 - a e^{j\omega_0} e^{-j\omega}} + \frac{1}{2} e^{-j\phi} \frac{1}{1 - a e^{-j\omega_0} e^{-j\omega}} \\ &= \frac{e^{j\phi}}{2} \frac{1}{1 - a e^{-j(\omega - \omega_0)}} + \frac{e^{-j\phi}}{2} \frac{1}{1 - a e^{-j(\omega + \omega_0)}} \end{aligned}$$

which is the same result as before.

DTFT Example 2

Suppose

$$x[n] = \begin{cases} a^n & n_1 \le n \le n_2 \\ 0 & \text{otherwise} \end{cases}$$

with integer $n_1 < n_2$. Find $X(e^{j\omega})$.

Note that we can write

$$\begin{aligned} x[n] &= a^n u[n-n_1] - a^n u[n-(n_2+1)] \\ &= a^{n_1} a^{n-n_1} u[n-n_1] - a^{n_2+1} a^{n-(n_2+1)} u[n-(n_2+1)] \end{aligned}$$

Hence, using linearity and the time-shifting property, we have

$$\begin{aligned} X(e^{j\omega}) &= a^{n_1} \frac{e^{-j\omega n_1}}{1-ae^{-j\omega}} - a^{n_2+1} \frac{e^{-j\omega(n_2+1)}}{1-ae^{-j\omega}} \\ &= \frac{(ae^{-j\omega})^{n_1} - (ae^{-j\omega})^{n_2+1}}{1-ae^{-j\omega}} \end{aligned}$$

Check symmetry properties for $a = 1$ and $n_1 = -n_2$...

Inverse DTFT Example 1

Suppose

$$X(e^{j\omega}) = 1 + 2\sum_{k=0}^{N}\cos(k\omega).$$

Find $x[n]$. One approach: Use Euler's identity to write

$$\begin{aligned} X(e^{j\omega}) &= 1 + 2\sum_{k=0}^{N}\left(\frac{1}{2}e^{jk\omega} + \frac{1}{2}e^{-jk\omega}\right) \\ &= 1 + \sum_{k=0}^{N}\left(e^{jk\omega} + e^{-jk\omega}\right) = 2 + \sum_{k=-N}^{N} e^{-jk\omega} \end{aligned}$$

Now use linearity, the time-shifting property, and the transform pair table to write

$$x[n] = 2\delta[n] + \sum_{k=-N}^{N}\delta[n-k] = \begin{cases} 3 & k=0 \\ 1 & k \in \{-N,\dots,N\}\backslash 0 \\ 0 & \text{otherwise.} \end{cases}$$

Inverse DTFT Example 2

Suppose

$$X(e^{j\omega}) = \frac{1}{(1 - ae^{-j\omega})^2}$$

with $|a| < 1$. Find $x[n]$. Observe that

$$j\frac{\partial}{\partial\omega}\left(\frac{1}{1 - ae^{-j\omega}}\right) = \frac{ae^{-j\omega}}{(1 - ae^{-j\omega})^2}$$

Hence, using the derivative in frequency property and the transform table, we can say

$$\frac{ae^{-j\omega}}{(1 - ae^{-j\omega})^2} \leftrightarrow na^n u[n]$$

But note that

$$\frac{ae^{-j\omega}}{(1 - ae^{-j\omega})^2} + \frac{1}{1 - ae^{-j\omega}} = \frac{1}{(1 - ae^{-j\omega})^2} = X(e^{j\omega})$$

Hence, by linearity, we have

$$x[n] = na^n u[n] + a^n u[n] = (n+1)a^n u[n].$$