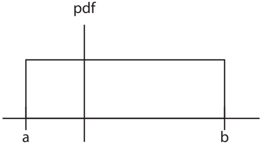
Digital Signal Processing
Discrete-Time Random Signals

D. Richard Brown III

Random Signal Basics (part 1 of 2)

Rather than mathematically specifying each sample of a discrete-time sequence $\{x[n]\}$, we can specify the sequence in terms of its **statistics**.

For instance, we can say $x[n]$ is **uniformly distributed** for all n on the interval $[a, b]$. This means that each $x[n]$ has the density



While the samples are random, the statistics are not. Here, the mean

$$m_x[n] = \mathrm{E}[x[n]] = \frac{a+b}{2}$$

and the variance

$$\mathrm{var}(x[n]) = \mathrm{E}[(x[n] - m_x)^2] = \frac{(a-b)^2}{12}.$$

Random Signal Basics (part 2 of 2)

It isn't enough to just specify the statistics of the individual samples. In general, we need to also know the **joint** statistics of multiple samples. For example, the autocorrelation

$$\phi_{xx}[n, n+m] = \mathrm{E}[x[n]x[n+m]]$$

is a measure of how samples at different times are related.

We will focus on a class of random signals called "wide sense stationary" (WSS) signals. These signals have

- Constant mean, i.e., $\mathrm{E}[x[n]] = m_x$.
- An autocorrelation function that is only a function of the time difference, i.e., $\mathrm{E}[x[n]x[n+m]] = \phi_{xx}[m]$.

Note that the average power of $x[n]$ is given as $\phi_{xx}[0] = \mathrm{E}[x^2[n]]$.

Mean of the Output of a DT-LTI System

Given a WSS input signal $x[n]$ with mean m_x, let's compute the mean of the output of a DT-LTI system

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k].$$

Note that $h[k]$ is not random. We can compute

$$\begin{aligned} m_y[n] &= \mathrm{E}\left[\sum_{k=-\infty}^{\infty} h[k]x[n-k]\right] \\ &= \mathrm{E}\left[\sum_{k=-\infty}^{\infty} h[k]x[n-k]\right] \\ &= \sum_{k=-\infty}^{\infty} h[k]\mathrm{E}\left[x[n-k]\right] \\ &= \sum_{k=-\infty}^{\infty} h[k]m_x = H(e^{j0})m_x \end{aligned}$$

Autocorrelation of the Output of a DT-LTI System

With the same setup, we can also compute the output autocorrelation:

$$
\begin{aligned}
\phi_{yy}[m] &= \mathrm{E}\left[y[n]y[n+m]\right] \\
&= \mathrm{E}\left[\left(\sum_{k=-\infty}^{\infty} h[k]x[n-k]\right)\left(\sum_{r=-\infty}^{\infty} h[r]x[n+m-r]\right)\right] \\
&= \mathrm{E}\left[\sum_{k=-\infty}^{\infty}\sum_{r=-\infty}^{\infty} h[k]h[r]x[n-k]x[n+m-r]\right] \\
&= \sum_{k=-\infty}^{\infty}\sum_{r=-\infty}^{\infty} h[k]h[r]\mathrm{E}\left[x[n-k]x[n+m-r]\right] \\
&= \sum_{k=-\infty}^{\infty}\sum_{r=-\infty}^{\infty} h[k]h[r]\phi_{xx}[m+k-r] \\
&\overset{\ell=r-k}{=} \sum_{\ell=-\infty}^{\infty} \phi_{xx}[m-\ell] \sum_{k=-\infty}^{\infty} h[k]h[k+\ell] = \sum_{\ell=-\infty}^{\infty} \phi_{xx}[m-\ell]c_{hh}[\ell]
\end{aligned}
$$

From this result and the prior result on the mean of the output process, we see that the output signal $y[n]$ is also WSS.

Power Spectral Density

For WSS signals, we can also consider the DTFT of the autocorrelation function

$$\Phi_{xx}(e^{j\omega}) = \sum_{m=-\infty}^{\infty} \phi_{xx}[m]e^{j\omega m}$$

From the prior result and the convolution property of the DTFT, we can relate the output power spectral density to the input power spectral density as

$$\begin{aligned}\Phi_{yy}(e^{j\omega}) &= C_{hh}(e^{j\omega})\Phi_{xx}(e^{j\omega}) \\ &= |H(e^{j\omega})|^2\Phi_{xx}(e^{j\omega})\end{aligned}$$

since $c_{hh}[n] = h[n] * h[-n]$.

Common Application: Filtering White Noise

White noise has $m_x = 0$ and $\phi_{xx}[m] = \sigma_x^2 \delta[m]$. It follows then that

$$\Phi_{xx}(e^{j\omega}) = \sum_{m=-\infty}^{\infty} \phi_{xx}[m] e^{j\omega m} = \sigma_x^2$$

for all ω.

Suppose this noise is filtered by a DT-LTI system with impulse response $h[n] = a^n u[n]$ with $|a| < 1$. Then

$$H(e^{j\omega}) = \frac{1}{1 - ae^{-j\omega}}$$

and

$$\Phi_{yy}(e^{j\omega}) = |H(e^{j\omega})|^2 \Phi_{xx}(e^{j\omega}) = \left| \frac{1}{1 - ae^{-j\omega}} \right|^2 \sigma_x^2 = \frac{\sigma_x^2}{1 + a^2 - 2a\cos(\omega)}.$$