Digital Signal Processing
IIR Filter Design via Impulse Invariance

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Basic Procedure

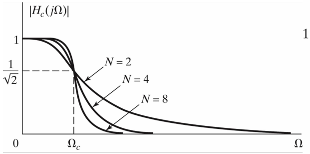
We assume here that we've already decided to use an IIR filter.

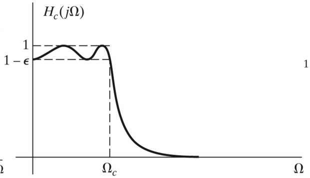
The basic procedure for IIR filter design via impulse invariance is:

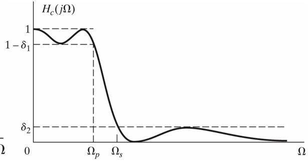
1. Determine the CT filter design method:
   1.1 Butterworth
   1.2 Chebychev Type I or Type II
   1.3 Elliptic
   1.4 ...
2. Transform the DT filter specifications to CT (sampling period T_d is arbitrary)
3. Design CT filter based on the magnitude squared response $|H_c(j\Omega)|^2$
   - Determine filter order
   - Determine cutoff frequency
4. Determine $H_c(s) \leftrightarrow h_c(t)$ corresponding to a stable causal filter
5. Convert to DT filter $H(z) \leftrightarrow h[n]$ via impulse invariance such that $h[n] = h_c(nT_d)$

Determining the Continuous-Time Filter Design Method

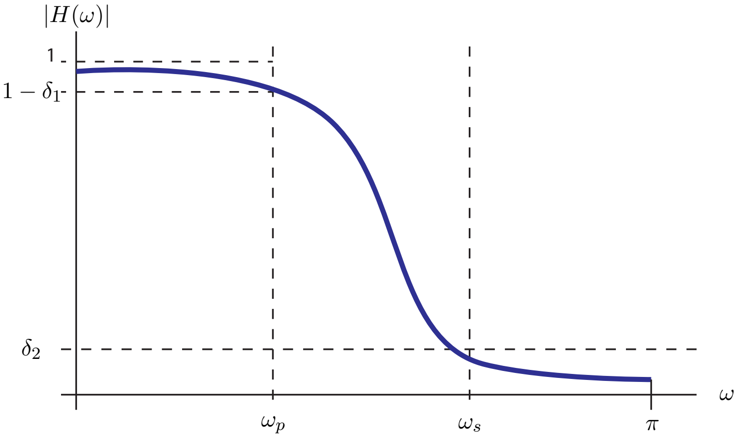
	passband	stopband
Butterworth	monotonic	monotonic
Chebychev Type I	equiripple	monotonic
Chebychev Type II	monotonic	equiripple
equiripple	equiripple	equiripple







Impulse-Invariant Lowpass Butterworth Filter Design Ex.



We start with the desired specifications of the DT filter. For this example, we will use $\omega_p = 0.2\pi$, $\omega_s = 0.3\pi$, $1 - \delta_1 = 0.89125$, and $\delta_2 = 0.17783$.

Convert DT Filter Specs to CT Filter Specs

We will use the Butterworth filter approach in this example. A CT Butterworth filter has a squared magnitude response given by

$$|H_c(j\Omega)|^2 = \frac{1}{1 + \left(\frac{j\Omega}{j\Omega_c}\right)^{2N}} \tag{1}$$

where Ω_c is the cutoff frequency (radians/second) and N is the filter order.

When designing filters via impulse invariance, the sampling period T_d is arbitrary. It is often convenient to just set $T_d = 1$ so that $\Omega = \omega$.

This implies the CT filter specs can be written as $\Omega_p = 0.2\pi$ and $\Omega_s = 0.3\pi$. Along with our magnitude specifications $1 - \delta_1 = 0.89125$ and $\delta_2 = 0.17783$, we can substitute these results directly into (1) to write

$$1 + \left(\frac{0.2\pi}{\Omega_c}\right)^{2N} \leq \left(\frac{1}{0.89125}\right)^2$$

$$1 + \left(\frac{0.3\pi}{\Omega_c}\right)^{2N} \geq \left(\frac{1}{0.17783}\right)^2$$

Determine the N and Ω_c

We can take the previous inequalities and write them as equalities as

$$1 + \left(\frac{0.2\pi}{\Omega_c}\right)^{2N} = \left(\frac{1}{0.89125}\right)^2 \quad (2)$$

$$1 + \left(\frac{0.3\pi}{\Omega_c}\right)^{2N} = \left(\frac{1}{0.17783}\right)^2. \quad (3)$$

We have two equations and two unknowns. By taking logarithms, we can isolate N and Ω_c to get

$$N = 5.8858$$

$$\Omega_c = 0.70474$$

Note that N must be an integer, so we can choose $N = 6$. We now can decide whether to pick Ω_c to match the passband spec (and exceed the stopband spec) or match the stopband spec (and exceed the passband spec). To minimize the effect of aliasing, we usually choose the former.

Determine $H_c(s)$ (part 1 of 3)

Given $N = 6$ and our choice to match the passband spec, we have the equality

$$1 + \left(\frac{0.2\pi}{\Omega_c}\right)^{12} = \left(\frac{1}{0.89125}\right)^2 \tag{4}$$

which gives $\Omega_c = 0.7032$. Now, to determine $H_c(s)$, we can write

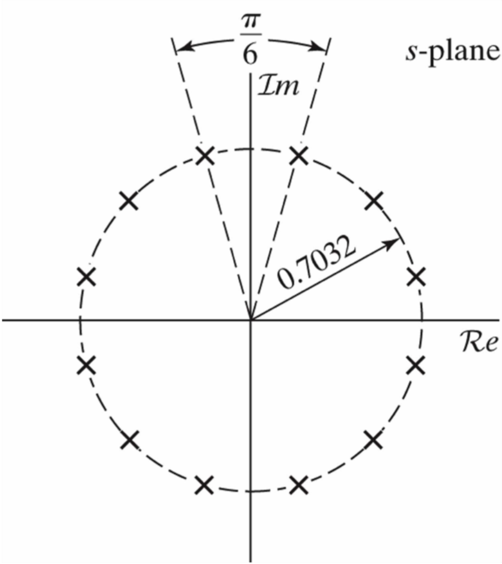
$$H_c(s)H_c(-s) = \frac{1}{1 + \left(\frac{s}{j\Omega_c}\right)^{2N}}.$$

The pole locations of $H_c(s)H_c(-s)$ follow from the fact that $1 + \left(\frac{x}{a}\right)^M = 0$ has M solutions given by

$$x = \{ae^{j\pi/M}, ae^{j3\pi/M}, \dots, ae^{j(\pi+(M-1)2\pi)/M}\}.$$

The $\frac{M}{2}$ poles corresponding to $H_c(s)$ are those in the left half plane.

Determine $H_c(s)$ (part 2 of 3)



Determine $H_c(s)$ (part 3 of 3)

Choosing the poles from the left half plane and doing a little bit of algebra, we can write

$$H_c(s) = \frac{0.12093}{(s^2 + 0.3640s + 0.4945)(s^2 + 0.9945s + 0.4945)(s^2 + 1.3585s + 0.4945)}$$

Since all of the poles are simple (non-repeated), we can write the partial fraction expansion

$$H_c(s) = \sum_{k=1}^{6} \frac{A_k}{s - s_k}$$

which implies that

$$h_c(t) = \begin{cases} \sum_{k=1}^{6} A_k e^{s_k t} & t \geq 0 \\ 0 & t < 0. \end{cases}$$

Determine $H(z)$ via Impulse Invariance

Impulse invariance requires $h[n] = h_c(nT_d)$ where we have previously chosen $T_d = 1$. Hence we have

$$h[n] = \begin{cases} \sum_{k=1}^{6} A_k e^{s_k n} & n \geq 0 \\ 0 & t < 0. \end{cases}$$

and it follows that

$$H(z) = \sum_{k=1}^{6} \frac{A_k}{1 - e^{s_k} z^{-1}}$$

with ROC $|z| >$ largest magnitude pole. A bit of algebra yields the final result

$$H(z) = \frac{0.2871 - 0.4466z^{-1}}{1 - 1.2971z^{-1} + 0.6949z^{-2}} + \frac{-2.1428 + 1.1455z^{-1}}{1 - 1.0691z^{-1} + 0.3699z^{-2}} + \frac{1.8577 - 0.6303z^{-1}}{1 - 0.9972z^{-1} + 0.2570z^{-2}}$$

which can immediately be realized in parallel form or rearranged to be realized in cascade or direct forms.

Impulse-Invariant Lowpass Butterworth Filter Design Ex.