Digital Signal Processing
IIR Filter Design via Bilinear Transform

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Basic Procedure

We assume here that we've already decided to use an IIR filter.

The basic procedure for IIR filter design via bilinear transform is:

1. Determine the CT filter class:
   1.1 Butterworth
   1.2 Chebychev Type I or Type II
   1.3 Elliptic
   1.4 ...
2. Transform the DT filter specifications to CT (sampling period T_d is arbitrary) **including prewarping the band edge frequencies**
3. Design CT filter based on the magnitude squared response $|H_c(j\Omega)|^2$
   - Determine filter order
   - Determine cutoff frequency
4. Determine $H_c(s)$ corresponding to a stable causal filter
5. Convert to DT filter $H(z)$ via bilinear transform such that

$$H(z) = H_c\left(\frac{2}{T_d}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\right)$$

Bilinear Transform

Idea: Given a causal stable LTI CT filter $H_c(s)$, we simply substitute

$$\boxed{s = \frac{2}{T_d}\left(\frac{1 - z^{-1}}{1 + z^{-1}}\right)}$$

to get $H(z)$.

This substitution is based on converting $H_c(s)$ to a differential equation, performing trapezoidal numerical integration with step size T_d to get a difference equation, and then converting the difference equation to a transfer function $H(z)$.

Remarks:

- Direct method to go from $H_c(s)$ to $H(z)$ that always works without going through the time-domain.
- This is a one-to-one mapping between points in the s-plane and z-plane.
- Points in the left-half (right-half) s-plane are mapped to points inside (outside) the unit circle on the z-plane.
- There is no aliasing even if $H_c(s)$ is not bandlimited.

Bilinear Transform: Simple Example

Suppose you are given a causal LTI CT system with $H_c(s) = \frac{1}{s-a}$. We can compute $H(z)$ straightforwardly with a little algebra:

$$
\begin{aligned}
H(z) &= H_c(s)\big|_{s=\frac{2}{T_d}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)} \\
&= \frac{1}{\frac{2}{T_d}\left(\frac{1-z^{-1}}{1+z^{-1}}\right) - a} \\
&= \frac{T_d(1+z^{-1})}{2(1-z^{-1}) - aT_d(1+z^{-1})} \\
&= \frac{T_d(1+z^{-1})}{(2-aT_d) - (2+aT_d)z^{-1}} \\
&= \frac{\beta(1+z^{-1})}{1-\alpha z^{-1}} \text{ (bilinar transform)}
\end{aligned}
$$

This is quite different than the $H(z)$ we computed via impulse invariance:

$$
H(z) = \frac{1}{1-e^{aT_d}z^{-1}} \text{ (impulse invariance)}
$$

Bilinear Transform: Frequency Response

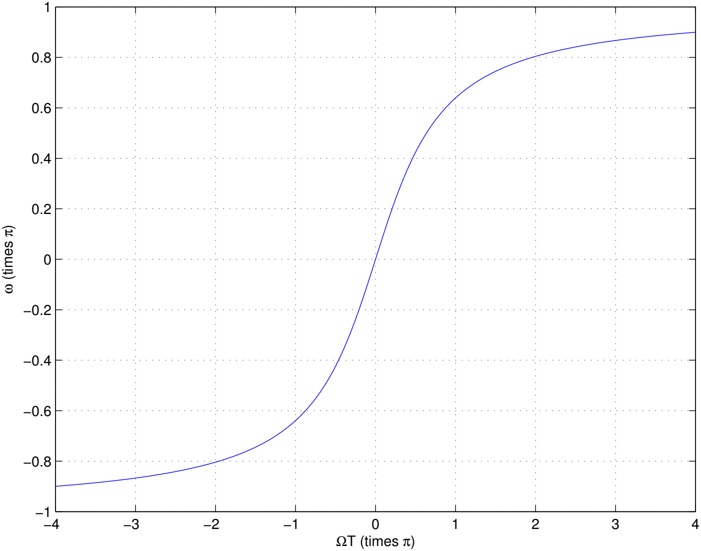
Given a causal stable LTI CT filter $H_c(s)$, we can compute $H(z)$ with via the bilinear transform. What is the relationship between $H_c(j\Omega)$ and $H(e^{j\omega})$?

Recall that $H_c(j\Omega) = H_c(s)|_{s=j\Omega}$ and $H(e^{j\omega}) = H(z)|_{z=e^{j\omega}}$. Substituting these into the bilinear transform formula, we get

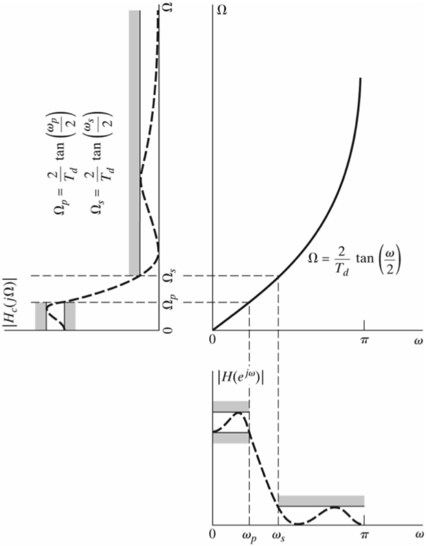
$$
\begin{aligned}
j\Omega &= \frac{2}{T_d}\left(\frac{1-e^{-j\omega}}{1+e^{-j\omega}}\right) \\
&= \frac{2}{T_d}\left(\frac{e^{j\omega/2}-e^{-j\omega/2}}{e^{j\omega/2}+e^{-j\omega/2}}\right) \\
&= \frac{2j}{T_d}\left(\frac{\frac{1}{2j}(e^{j\omega/2}-e^{-j\omega/2})}{\frac{1}{2}(e^{j\omega/2}+e^{-j\omega/2})}\right) \\
&= \frac{2j}{T_d}\tan(\omega/2)
\end{aligned}
$$

Hence $\Omega = \frac{2}{T_d}\tan(\omega/2)$ or $\omega = 2\tan^{-1}(\Omega T_d/2)$. This nonlinear relationship is called "frequency warping".

Bilinear Transform: Frequency Warping



Bilinear Transform: Consequences of Frequency Warping



The good news is that we don't have to worry about aliasing. The "bad" news is that we have to account for frequency warping when we start from a discrete-time filter specification.

$$\Omega_p = \frac{2}{T_d}\tan(\omega_p/2)$$
$$\Omega_s = \frac{2}{T_d}\tan(\omega_s/2)$$

Bilinear Transform Lowpass Butterworth Filter Design Ex.

We start with the desired specifications of the DT filter. Suppose $\omega_p = 0.2\pi$, $\omega_s = 0.3\pi$, $1 - \delta_1 = 0.89125$, and $\delta_2 = 0.17783$.

Our first step is to convert the DT filter specs to CT filter specs via the pre-warping equations. Setting $T_d = 1$, we can compute

$$\begin{aligned}\Omega_p &= \frac{2}{T_d}\tan(0.2\pi/2) = 0.6498\\ \Omega_s &= \frac{2}{T_d}\tan(0.3\pi/2) = 1.0191\end{aligned}$$

Assuming a Butterworth design, the squared magnitude response given by

$$|H_c(j\Omega)|^2 = \frac{1}{1 + \left(\frac{j\Omega}{j\Omega_c}\right)^{2N}} \tag{1}$$

We can substitute our specs directly into (1) to write

$$\begin{aligned}1 + \left(\frac{0.6498}{\Omega_c}\right)^{2N} &\le \left(\frac{1}{0.89125}\right)^2\\ 1 + \left(\frac{1.0191}{\Omega_c}\right)^{2N} &\ge \left(\frac{1}{0.17783}\right)^2\end{aligned}$$

Determine the N and Ω_c

We can take the previous inequalities and write them as equalities as

$$1 + \left(\frac{0.6498}{\Omega_c}\right)^{2N} = \left(\frac{1}{0.89125}\right)^2$$
$$1 + \left(\frac{1.0191}{\Omega_c}\right)^{2N} = \left(\frac{1}{0.17783}\right)^2$$

We have two equations and two unknowns. Solving for N, we get $N = 5.305$. Note that N must be an integer, so we can choose $N = 6$.

We now can decide whether to pick Ω_c to match the passband spec (and exceed the stopband spec) or match the stopband spec (and exceed the passband spec). Since we don't need to worry about aliasing when using bilinear transforms, we often choose the latter.

Plugging $N = 6$ into the second equality and solving for Ω_c yields $\Omega_c = 0.766$.

Bilinear Transform Lowpass Butterworth Filter Design Ex.

The remaining steps in deriving $H_c(s)$ are identical to those we saw when looking at impulse invariant filter design. By choosing the poles of $H_c(s)H_c(-s)$ in the left half plane, we have

$$H_c(s) = \frac{0.20238}{(s^2 + 0.3966s + 0.5871)(s^2 + 1.0836s + 0.5871)(s^2 + 1.4802s + 0.5871)}$$

We then substitute

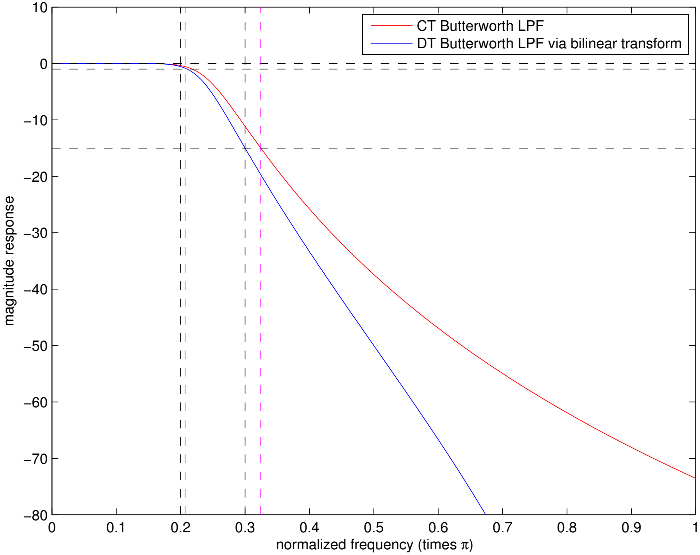
$$s = \frac{2}{T_d}\left(\frac{1 - z^{-1}}{1 + z^{-1}}\right)$$

to get the final result

$$H(z) = \frac{0.0007378(1 + z^{-1})^6}{(1 – 1.2686z^{-1} + 0.7051z^{-2})(1 – 1.0106z^{-1} + 0.3583z^{-2})(1 – 0.9044z^{-1} + 0.2155z^{-2})}$$

which can be implemented in any of the usual realization structures (direct, cascade, parallel, ...).

Bilinear Transform Lowpass Butterworth Filter Design Ex.



Bilinear Transform Lowpass Butterworth Filter Design Ex.