Digital Signal Processing
Introduction to the Discrete Fourier Transform

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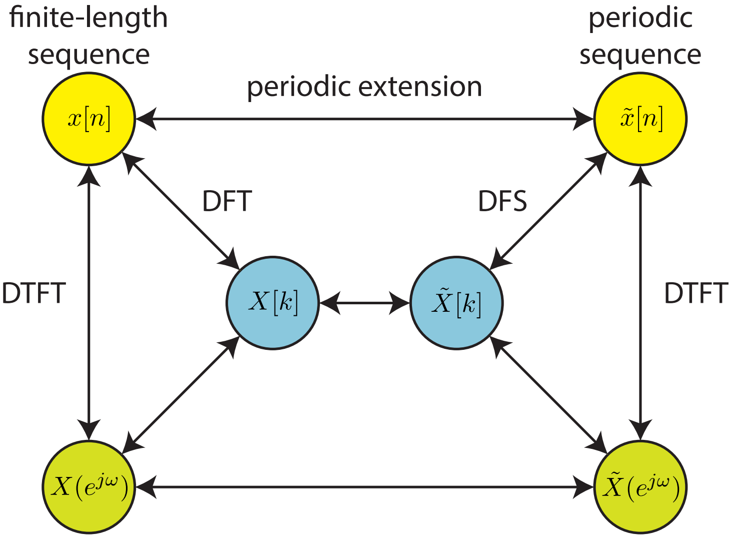
Big Picture

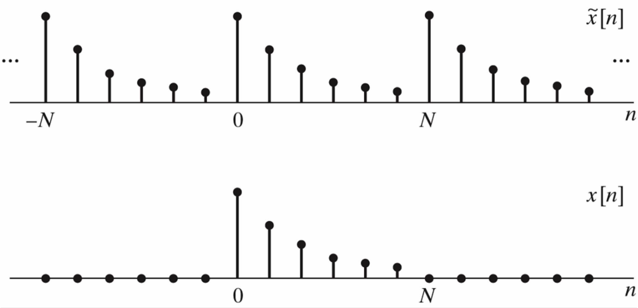

finite-length sequence
periodic sequence
periodic extension
$x[n]$
$\tilde{x}[n]$
DFT
DFS
DTFT
$X[k]$
$\tilde{X}[k]$
DTFT
$X(e^{j\omega})$
$\tilde{X}(e^{j\omega})$

Period Extensions of Finite-Length Sequences (part 1 of 2)

Given a causal finite-length sequence $x[n]$ equal to zero for all $n < 0$ and all $n > N-1$, we can form a periodic sequence $\tilde{x}[n]$

$$\tilde{x}[n] = \sum_{r=-\infty}^{\infty} x[n - rN] = x[((n))_N]$$

for all $n \in \mathbb{Z}$ where the notation $((n))_N$ means "n modulo N".



Period Extensions of Finite-Length Sequences (part 2 of 2)

Another way to construct a periodic sequence $\tilde{x}[n]$ from a finite-length sequence $x[n]$ is to convolve the $x[n]$ with a periodic impulse train $\tilde{p}[n]$ defined as

$$\tilde{p}[n] = \sum_{r=-\infty}^{\infty} \delta[n - rN]$$

It is easy to see that

$$\tilde{x}[n] = x[n] * \tilde{p}[n]$$

is equivalent to our previous construction.

The inverse operation (extracting a causal finite-length sequence from a periodic sequence) is also straightforward

$$x[n] = \begin{cases} \tilde{x}[n] & n = 0, 1, \dots, N-1 \\ 0 & \text{otherwise} \end{cases}$$

Review: DTFT of $x[n]$

To establish a fundamental relationship between finite-length sequences and periodic sequences, recall that the DTFT of a general sequence $x[n]$ is defined as

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}.$$

Recall that the DTFT converges in certain senses, e.g., uniformly, depending on certain condition on $x[n]$, e.g., $x[n]$ absolutely summable.

In the case when $x[n]$ is a causal finite-length sequence we can write

$$X(e^{j\omega}) = \sum_{n=0}^{N-1} x[n]e^{-j\omega n}$$

which converges uniformly as long as each element of the sequence is finite.

Summary of Main Results (part 1 of 2)

1. We will show that, since $\tilde{x}[n]$ is a periodic sequence, we can describe it in terms of a **discrete Fourier series** (DFS) such that

$$\tilde{x}[n] = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}[k] e^{j2\pi kn/N}$$

   where $\tilde{X}[k]$ is a periodic sequence containing the DFS coefficients.

2. The DTFT of $\tilde{x}[n]$ is not so simple since $\tilde{x}[n]$ is not absolutely summable or even square-summable (unless $x[n] = 0$). Nevertheless, we will show

$$\tilde{X}(e^{j\omega}) = \sum_{k=-\infty}^{\infty} \frac{2\pi}{N} X(e^{j2\pi k/N}) \delta(\omega - 2\pi k/N).$$

   where $X(e^{j2\pi k/N})$ is the DTFT of the finite-length sequence $x[n]$.

3. We will show the DFS coefficients can be related to the DTFT of $x[n]$ as

$$\tilde{X}[k] = X(e^{j2\pi k/N}) = X(e^{j\omega})|_{\omega=2\pi k/N}$$

Summary of Main Results (part 2 of 2)

4. We will define the DFT of a causal finite-length sequence as

$$X[k] = \sum_{n=0}^{N-1} x[n]e^{-j2\pi kn/N}$$

   for $k = 0, \dots, N-1$.

5. We will show

$$X[k] = \begin{cases} \tilde{X}[k] & n = 0, \dots, N-1 \\ 0 & \text{otherwise.} \end{cases}$$

   This result along with result 3 implies

$$X[k] = \begin{cases} X(e^{j2\pi k/N}) & n = 0, \dots, N-1 \\ 0 & \text{otherwise.} \end{cases}$$

   which shows that the DFT of a finite-length sequence $x[n]$ is a uniformly sampled version of the DTFT of $x[n]$ for $\omega = 0, \frac{2\pi}{N}, \dots, \frac{2\pi(N-1)}{N}$.