Digital Signal Processing
The Discrete Fourier Series of Periodic Sequences

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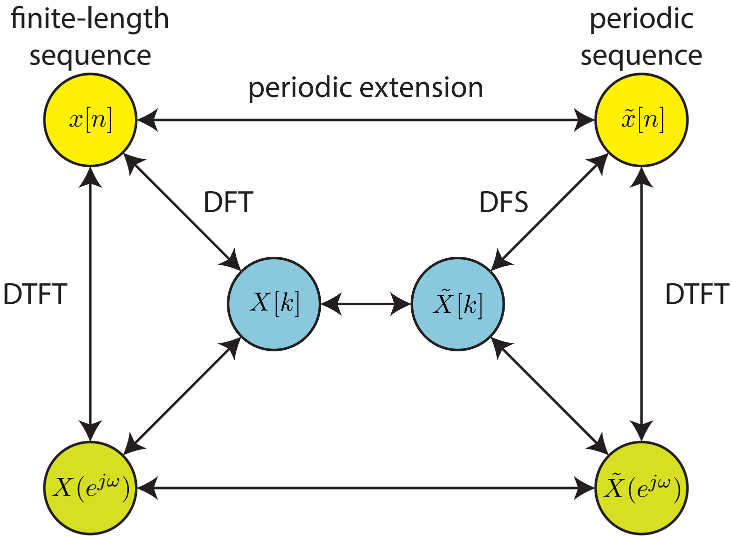
Big Picture


finite-length sequence
periodic sequence
periodic extension
$x[n]$
$\tilde{x}[n]$
DFT
DFS
DTFT
$X[k]$
$\tilde{X}[k]$
DTFT
$X(e^{j\omega})$
$\tilde{X}(e^{j\omega})$

Review: Fourier Series of Continuous-Time Periodic Signals

Suppose we have a periodic continuous-time signal $\tilde{x}(t)$ with period T_0 such that $\tilde{x}(t + rT_0) = \tilde{x}(t)$ for all t and all integer r. We denote $\Omega_0 = \frac{2\pi}{T_0}$ as the radian frequency corresponding to the period T_0.

Under certain conditions satisfied for most signals of interest in signal processing, such a periodic signal can be expressed as a sum of complex exponentials with frequency $0, \Omega_0, 2\Omega_0, 3\Omega_0, \dots$, i.e.,

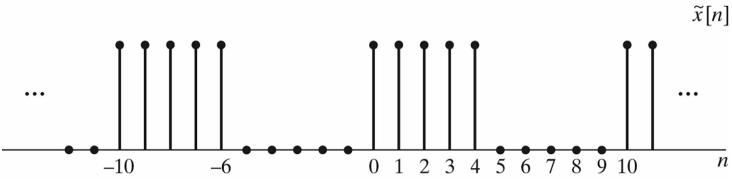
$$\tilde{x}(t) = \frac{1}{T_0} \sum_{k=-\infty}^{\infty} C[k] e^{j\Omega_0 kt}$$

where $C[k]$ are the (usually complex) Fourier series coefficients. These coefficients can be easily computed by observing that the set of functions $\{e^{j\Omega_0 kt}\}$ for $k \in \mathbb{Z}$ is an orthogonal basis, leading to

$$C[k] = \int_{T_0} \tilde{x}(t) e^{-j\Omega_0 kt}\, dt$$

Basis Functions for the Discrete Fourier Series (DFS)

The principle of the discrete Fourier series (DFS) is very similar to the continuous-time case. We assume a periodic discrete-time sequence $\tilde{x}[n]$ with period N samples.



We denote $\omega_0 = \frac{2\pi}{N}$ as the normalized frequency corresponding to the period N and consider the set of discrete-time periodic complex exponential sequences

$$\{e^{j\omega_0 kn}\} = \{e^{j2\pi kn/N}\} = \{e_k[n]\}$$

with $k \in \mathbb{Z}$ and $n \in \mathbb{Z}$.

The DFS is a Finite Sum

Observe that, for integer r, we have

$$e_{k+rN}[n] = e^{j2\pi(k+rN)n/N} = e^{j2\pi kn/N} e^{j2\pi rnN/N} = e^{j2\pi kn/N} = e_k[n]$$

hence our original set $\{e_k[n]\}$ for $k \in \mathbb{Z}$ has many redundant elements. In fact, there are only N distinct periodic complex exponentials in the set, which we can choose as

$$\{e_0[n], e_1[n], \dots, e_{N-1}[n]\}$$

for $n \in \mathbb{Z}$.

Along the same lines as the continuous-time Fourier series, we can then write the DFS

$$\tilde{x}[n] = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}[k] e_k[n] = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}[k] e^{j2\pi kn/N}$$

where $\tilde{X}[k]$ are the (usually complex) DFS coefficients. A key difference with respect to the continuous-time Fourier series is that the DFS of a periodic sequence $\tilde{x}[n]$ can be written as a **finite sum**.

Computing the DFS Coefficients $\tilde{X}[k]$

We have

$$\tilde{x}[n] = \frac{1}{N}\sum_{r=0}^{N-1}\tilde{X}[r]e_k[r]$$

If we multiply both sides by $e_k^*[n]$ and sum from $n = 0, \dots, N-1$ we can write

$$\sum_{n=0}^{N-1}\tilde{x}[n]e_k^*[n] = \sum_{n=0}^{N-1}\frac{1}{N}\sum_{r=0}^{N-1}\tilde{X}[r]e_r[n]e_k^*[n] = \sum_{r=0}^{N-1}\tilde{X}[r]\left(\frac{1}{N}\sum_{n=0}^{N-1}e_r[n]e_k^*[n]\right).$$

Note that

$$\frac{1}{N}\sum_{n=0}^{N-1}e_r[n]e_k^*[n] = \frac{1}{N}\sum_{n=0}^{N-1}e^{j2\pi(r-k)n} = \begin{cases}1 & r-k = mN \text{ with } m \text{ an integer}\\ 0 & \text{otherwise.}\end{cases}$$

Hence

$$\sum_{n=0}^{N-1}\tilde{x}[n]e_k^*[n] = \tilde{X}[k]$$

which is similar to the CT case with an integral is over one period of $\tilde{x}(t)e^{-j\Omega_0 kt}$.

Some Properties of the DFS

TABLE 8.1 SUMMARY OF PROPERTIES OF THE DFS

	Periodic Sequence (Period N)	DFS Coefficients (Period N)
1.	$\tilde{x}[n]$	$\tilde{X}[k]$ periodic with period N
2.	$\tilde{x}_1[n], \tilde{x}_2[n]$	$\tilde{X}_1[k], \tilde{X}_2[k]$ periodic with period N
3.	$a\tilde{x}_1[n] + b\tilde{x}_2[n]$	$a\tilde{X}_1[k] + b\tilde{X}_2[k]$
4.	$\tilde{X}[n]$	$N\tilde{x}[-k]$
5.	$\tilde{x}[n-m]$	$W_N^{km}\tilde{X}[k]$
6.	$W_N^{-\ell n}\tilde{x}[n]$	$\tilde{X}[k-\ell]$
7.	$\sum_{m=0}^{N-1} \tilde{x}_1[m]\tilde{x}_2[n-m]$ (periodic convolution)	$\tilde{X}_1[k]\tilde{X}_2[k]$
8.	$\tilde{x}_1[n]\tilde{x}_2[n]$	$\frac{1}{N}\sum_{\ell=0}^{N-1} \tilde{X}_1[\ell]\tilde{X}_2[k-\ell]$ (periodic convolution)
9.	$\tilde{x}^*[n]$	$\tilde{X}^*[-k]$
10.	$\tilde{x}^*[-n]$	$\tilde{X}^*[k]$

Periodic Convolution Property of the DFS

Recall, if we had signals $x_i[n]$ with the necessary DTFTs we could write

$$x_3[n] = \sum_{k=-\infty}^{\infty} x_1[k]x_2[n-k] \quad \Leftrightarrow \quad X_3(e^{j\omega}) = X_1(e^{j\omega})X_2(e^{j\omega}).$$

For the DFS, we have have this relationship only for **periodic convolution**:

$$\tilde{x}_3[n] = \sum_{k=0}^{N-1} \tilde{x}_1[k]\tilde{x}_2[n-k] \quad \Leftrightarrow \quad \tilde{X}_3[k] = \tilde{X}_1[k]\tilde{X}_2[k].$$

Remarks:

1. If $\tilde{x}_1[n]$ and $\tilde{x}_2[n]$ are both periodic with period N, then so is $\tilde{x}_3[n]$.
2. Periodic convolution specifies a sum over one period.
3. Since all of the signal are periodic, $\tilde{x}_2[n-k] = \tilde{x}_2[n-k+mN]$ for integer M. For example $\tilde{x}_2[1-3] = \tilde{x}_2[-2] = \tilde{x}_2[N-2]$.

Periodic Convolution Example

Suppose the periodic sequences $\tilde{x}_1[n]$ and $\tilde{x}_2[n]$ are generated as periodic extensions of

$$x_1[n] = \{\underline{1}, 2, 3\}$$
$$x_2[n] = \{\underline{a}, b, c\}$$

with $N = 3$. Then the periodic sequence $\tilde{x}_3[n]$ can be computed as

$$\tilde{x}_3[0] = \sum_{k=0}^{2} \tilde{x}_1[k]\tilde{x}_2[0-k] = 1 \cdot a + 2 \cdot c + 3 \cdot b$$

$$\tilde{x}_3[1] = \sum_{k=0}^{2} \tilde{x}_1[k]\tilde{x}_2[1-k] = 1 \cdot b + 2 \cdot a + 3 \cdot c$$

$$\tilde{x}_3[2] = \sum_{k=0}^{2} \tilde{x}_1[k]\tilde{x}_2[2-k] = 1 \cdot c + 2 \cdot b + 3 \cdot a$$

and all other elements of $\tilde{x}_3[n]$ are implied through the periodicity.