Digital Signal Processing
The DTFT of a Periodic Sequence and its Relation to the DFS

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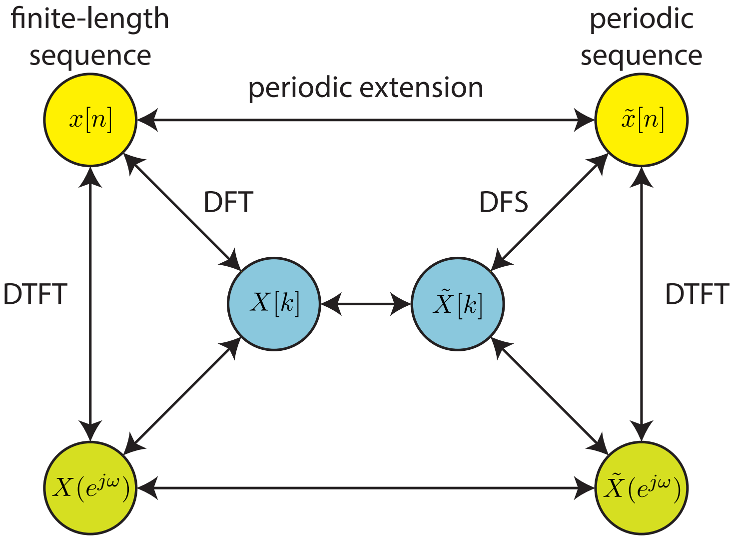
Big Picture


finite-length sequence
periodic sequence
periodic extension
$x[n]$
$\tilde{x}[n]$
DFT
DFS
DTFT
$X[k]$
$\tilde{X}[k]$
DTFT
$X(e^{j\omega})$
$\tilde{X}(e^{j\omega})$

Review: DTFT of $x[n] = 1$

Recall that the sequence

$$x[n] = 1 \quad \forall n$$

doesn't have absolute summability or squared summability, hence the DTFT summation does not converge in any of the usual senses. We can however "guess" at the DTFT as

$$X(e^{j\omega}) = 2\pi \sum_{r=-\infty}^{\infty} \delta(\omega - 2\pi r)$$

and compute the inverse DTFT

$$\begin{aligned} x[n] &= \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n}\, d\omega \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} 2\pi \delta(\omega) e^{j\omega n}\, d\omega \\ &= 1 \quad \forall n \end{aligned}$$

to confirm this "guess" is the correct DTFT.

DTFT of Periodic Impulse Train

Now consider the periodic discrete-time impulse train sequence

$$\tilde{p}[n] = \sum_{r=-\infty}^{\infty} \delta[n - rN].$$

Observe that this is like $x[n] = 1$ upsampled by N, i.e., there are $N-1$ zeros inserted between each pair of samples in $x[n]$. Recall that integer upsampling by a factor of N just squeezes the spectrum such that

$$\begin{aligned}\tilde{P}(e^{j\omega}) &= X(e^{j\omega N}) \\ &= 2\pi \sum_{r=-\infty}^{\infty} \delta(\omega N - 2\pi r) \\ &= \frac{2\pi}{N} \sum_{r=-\infty}^{\infty} \delta\left(\omega - \frac{2\pi r}{N}\right)\end{aligned}$$

where the last equality is from the scaling property of the Dirac delta function: $\delta(\alpha t) = \alpha^{-1}\delta(t)$.

DTFT of Periodic Signal

This last result allows us to write an expression for the DTFT of a periodic signal

$$\tilde{x}[n] = \tilde{p}[n] * x[n]$$

where $x[n]$ is a finite-length signal with length N and DTFT $X(e^{j\omega})$.

We can write

$$\begin{aligned}
\tilde{X}(e^{j\omega}) &= \tilde{P}(e^{j\omega})X(e^{j\omega}) \\
&= \frac{2\pi}{N}\sum_{r=-\infty}^{\infty} X(e^{j\omega})\delta\left(\omega - \frac{2\pi r}{N}\right) \\
&= \frac{2\pi}{N}\sum_{r=-\infty}^{\infty} X(e^{j2\pi r/N})\delta\left(\omega - \frac{2\pi r}{N}\right)
\end{aligned}$$

Observe that $\tilde{X}(e^{j\omega})$ is just a periodic series of impulses with weights given as $X(e^{j\omega})$ for $\omega = 2\pi r/N$ for $r \in \mathbb{Z}$.

Relation Between $\tilde{X}(e^{j\omega})$ and $\tilde{X}[k]$

We have the result

$$\tilde{X}(e^{j\omega}) = \frac{2\pi}{N} \sum_{r=-\infty}^{\infty} X(e^{j2\pi r/N})\delta\left(\omega - \frac{2\pi r}{N}\right)$$

with

$$X(e^{j2\pi r/N}) = \sum_{n=0}^{N-1} x[n]e^{-j2\pi rn/N} = \sum_{n=0}^{N-1} \tilde{x}[n]e^{-j2\pi rn/N} = \tilde{X}[r].$$

Hence, we can write

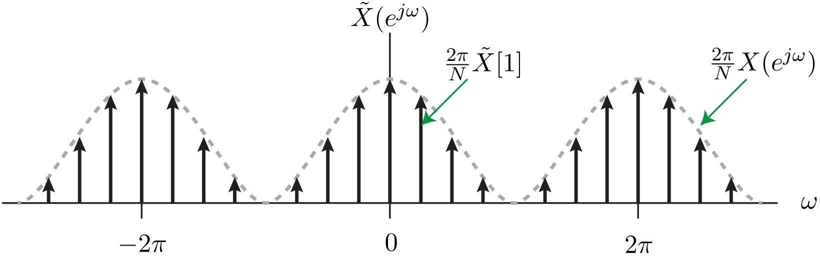
$$\tilde{X}(e^{j\omega}) = \frac{2\pi}{N} \sum_{r=-\infty}^{\infty} \tilde{X}[r]\delta\left(\omega - \frac{2\pi r}{N}\right)$$

Example

We have

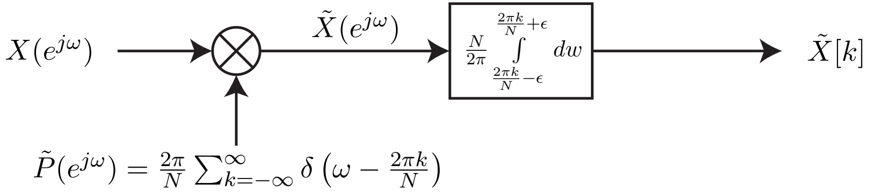
$$\tilde{X}(e^{j\omega}) = \frac{2\pi}{N} \sum_{r=-\infty}^{\infty} X(e^{j\omega})\delta\left(\omega - \frac{2\pi r}{N}\right)$$
$$= \frac{2\pi}{N} \sum_{r=-\infty}^{\infty} \tilde{X}[r]\delta\left(\omega - \frac{2\pi r}{N}\right)$$

which, for example, can be drawn as



Interpretation: Ideal Sampling of the DTFT

The relationship between $X(e^{j\omega})$, $\tilde{X}(e^{j\omega})$, and $\tilde{X}[k]$ identical to what we saw earlier for ideal sampling. The difference here is that the sampling is performed on the DTFT $X(e^{j\omega})$.



Hence, the DFS coefficients $\tilde{X}[k]$ of the periodic sequence $\tilde{x}[n]$ correspond to an ideal sampling of the DTFT $X(e^{j\omega})$ of the aperiodic (length N) sequence $x[n]$ at frequencies $\omega = \frac{2\pi k}{N}$.

Time-Domain Aliasing (part 1 of 2)

As an example, suppose $x[n] = \{\underline{0.5}, 1, 0.5\}$ and note that

$$X(e^{j\omega}) = e^{-j\omega}(1 + \cos\omega).$$

What happens if we sample this DTFT with $N = 2$?

In this case, we get

$$\tilde{X}[0] = X(e^{j0}) = 2 \qquad \text{and} \qquad \tilde{X}[1] = X(e^{j\pi}) = 0$$

and the inverse DFS yields

$$\begin{aligned} \tilde{x}[0] &= \frac{1}{2}\left(\tilde{X}[0]e^{j0} + \tilde{X}[1]e^{j0}\right) = 1 \\ \tilde{x}[1] &= \frac{1}{2}\left(\tilde{X}[0]e^{j0} + \tilde{X}[1]e^{j\pi}\right) = 1. \end{aligned}$$

In fact, it is easy to confirm $\tilde{x}[n] = 1$ for all n, which is not a periodic extension of $x[n]$.

Time-Domain Aliasing (part 2 of 2)

Note that $\tilde{x}[n]$ is still periodic, it is just a length-two periodic extension of $x[n]$ rather than the usual length-N periodic extension of $x[n]$ (here $N = 3$).

Sampling the DTFT always induces a periodic sequence in the time domain (in the same way that sampling $x[n]$ always results in a periodic $X(e^{j\omega})$).

In this example, we have under sampled $X(e^{j\omega})$ and induced time-domain aliasing in the periodic signal $\tilde{x}[n]$.