Digital Signal Processing
The DTFT of a Periodic Sequence
and its Relation to the DFS

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DSP: The DTFT of a Periodic Sequence

Big Picture

finite-length sequence

$x[n]$  \[\xrightarrow{\text{periodic extension}}\] \[\tilde{x}[n]\]

periodic sequence

$X[k]$  \[\xrightarrow{\text{DFT}}\] \[\tilde{X}[k]\]

$X(e^{j\omega})$  \[\xrightarrow{\text{DTFT}}\] \[\tilde{X}(e^{j\omega})\]
Review: DTFT of $x[n] = 1$

Recall that the sequence

$$x[n] = 1 \quad \forall n$$

doesn’t have absolute summability or squared summability, hence the DTFT summation does not converge in any of the usual senses. We can however “guess” at the DTFT as

$$X(e^{j\omega}) = 2\pi \sum_{r=-\infty}^{\infty} \delta(\omega - 2\pi r)$$

and compute the inverse DTFT

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} \, d\omega$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} 2\pi \delta(\omega) e^{j\omega n} \, d\omega$$

$$= 1 \quad \forall n$$

to confirm this “guess” is the correct DTFT.
Now consider the periodic discrete-time impulse train sequence

\[ \tilde{p}[n] = \sum_{r=\infty}^{\infty} \delta[n - rN]. \]

Observe that this is like \( x[n] = 1 \) upsampled by \( N \), i.e., there are \( N - 1 \) zeros inserted between each pair of samples in \( x[n] \). Recall that integer upsampling by a factor of \( N \) just squeezes the spectrum such that

\[ \tilde{P}(e^{j\omega}) = X(e^{j\omega N}) \]

\[ = 2\pi \sum_{r=-\infty}^{\infty} \delta(\omega N - 2\pi r) \]

\[ = \frac{2\pi}{N} \sum_{r=-\infty}^{\infty} \delta \left( \omega - \frac{2\pi r}{N} \right) \]

where the last equality is from the scaling property of the Dirac delta function: \( \delta(\alpha t) = \alpha^{-1}\delta(t) \).
DTFT of Periodic Signal

This last result allows us to write an expression for the DTFT of a periodic signal

\[ \tilde{x}[n] = \tilde{p}[n] \ast x[n] \]

where \( x[n] \) is a finite-length signal with length \( N \) and DTFT \( X(e^{j\omega}) \).

We can write

\[
\tilde{X}(e^{j\omega}) = \tilde{P}(e^{j\omega})X(e^{j\omega}) \\
= \frac{2\pi}{N} \sum_{r=-\infty}^{\infty} X(e^{j\omega}) \delta \left( \omega - \frac{2\pi r}{N} \right) \\
= \frac{2\pi}{N} \sum_{r=-\infty}^{\infty} X(e^{j2\pi r/N}) \delta \left( \omega - \frac{2\pi r}{N} \right)
\]

Observe that \( \tilde{X}(e^{j\omega}) \) is just a periodic series of impulses with weights given as \( X(e^{j\omega}) \) for \( \omega = 2\pi r/N \) for \( r \in \mathbb{Z} \).
Relation Between $\tilde{X}(e^{j\omega})$ and $\tilde{X}[k]$ 

We have the result

$$\tilde{X}(e^{j\omega}) = \frac{2\pi}{N} \sum_{r=-\infty}^{\infty} X(e^{j2\pi r/N}) \delta \left( \omega - \frac{2\pi r}{N} \right)$$

with

$$X(e^{j2\pi r/N}) = \sum_{n=0}^{N-1} x[n] e^{-j2\pi rn/N} = \sum_{n=0}^{N-1} \tilde{x}[n] e^{-j2\pi rn/N} = \tilde{X}[r].$$

Hence, we can write

$$\tilde{X}(e^{j\omega}) = \frac{2\pi}{N} \sum_{r=-\infty}^{\infty} \tilde{X}[r] \delta \left( \omega - \frac{2\pi r}{N} \right)$$
We have

\[ \tilde{X}(e^{j\omega}) = \frac{2\pi}{N} \sum_{r=-\infty}^{\infty} X(e^{j\omega}) \delta \left( \omega - \frac{2\pi r}{N} \right) \]

\[ = \frac{2\pi}{N} \sum_{r=-\infty}^{\infty} \tilde{X}[r] \delta \left( \omega - \frac{2\pi r}{N} \right) \]

which, for example, can be drawn as

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\[ \tilde{X}(e^{j\omega}) \]

\[ \frac{2\pi}{N} \tilde{X}[1] \]

\[ \frac{2\pi}{N} X(e^{j\omega}) \]

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\[ \omega \]
The relationship between $X(e^{j\omega})$, $\tilde{X}(e^{j\omega})$, and $\tilde{X}[k]$ identical to what we saw earlier for ideal sampling. The difference here is that the sampling is performed on the DTFT $X(e^{j\omega})$.

$$\tilde{P}(e^{j\omega}) = \frac{2\pi}{N} \sum_{k=-\infty}^{\infty} \delta \left( \omega - \frac{2\pi k}{N} \right)$$

Hence, the DFS coefficients $\tilde{X}[k]$ of the periodic sequence $\tilde{x}[n]$ correspond to an ideal sampling of the DTFT $X(e^{j\omega})$ of the aperiodic (length $N$) sequence $x[n]$ at frequencies $\omega = \frac{2\pi k}{N}$. 
As an example, suppose \( x[n] = \{0.5, 1, 0.5\} \) and note that
\[
X(e^{j\omega}) = e^{-j\omega}(1 + \cos \omega).
\]
What happens if we sample this DTFT with \( N = 2 \)?

In this case, we get
\[
\tilde{X}[0] = X(e^{j0}) = 2 \quad \text{and} \quad \tilde{X}[1] = X(e^{j\pi}) = 0
\]
and the inverse DFS yields
\[
\tilde{x}[0] = \frac{1}{2} \left( \tilde{X}[0]e^{j0} + \tilde{X}[1]e^{j0} \right) = 1
\]
\[
\tilde{x}[1] = \frac{1}{2} \left( \tilde{X}[0]e^{j0} + \tilde{X}[1]e^{j\pi} \right) = 1.
\]

In fact, it is easy to confirm \( \tilde{x}[n] = 1 \) for all \( n \), which is not a periodic extension of \( x[n] \).

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Time-Domain Aliasing (part 2 of 2)

Note that $\tilde{x}[n]$ is still periodic, it is just a length-two periodic extension of $x[n]$ rather than the usual length-$N$ periodic extension of $x[n]$ (here $N = 3$).

Sampling the DTFT always induces a periodic sequence in the time domain (in the same way that sampling $x[n]$ always results in a periodic $X(e^{j\omega})$).

In this example, we have under sampled $X(e^{j\omega})$ and induced time-domain aliasing in the periodic signal $\tilde{x}[n]$. 