Digital Signal Processing
Non-Square Discrete Fourier Transforms

D. Richard Brown III

Length-M DFT of a Length-N Sequence

Recall a length-$N < \infty$ sequence $\{x[n]\}$ has the N-point DFT $\{X[k]\}$ given as

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j2\pi kn/N} = \sum_{n=0}^{N-1} x[n] W_N^{kn} \qquad \text{for } k = 0, \dots, N-1$$

where the "twiddle factors" $W_N := e^{-j2\pi/N}$. We have previously shown that that the DFT is just a sampled version of the DTFT such that

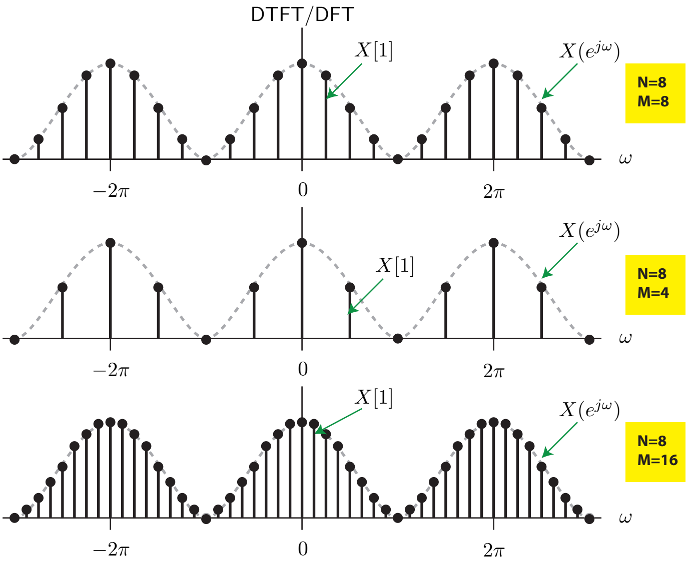
$$X[k] = X(e^{j\omega})|_{\omega=2\pi k/N} \qquad \text{for } k = 0, \dots, N-1.$$

We can generalize the DFT of a length-N sequence so that

$$X[k] = X(e^{j\omega})|_{\omega=2\pi k/M} \qquad \text{for } k = 0, \dots, M-1.$$

where M does not necessarily equal N. This just corresponds to a different (finer or coarser) sampling of the DTFT.

Example


DTFT/DFT
$X[1]$
$X(e^{j\omega})$
N=8
M=8
-2π
0
2π
ω
$X[1]$
$X(e^{j\omega})$
N=8
M=4
-2π
0
2π
ω
$X[1]$
$X(e^{j\omega})$
N=8
M=16
-2π
0
2π
ω

Case 1: $M > N$

When $M > N$, we are sampling the DTFT on a finer grid than the usual case when $M = N$. For the length-N sequence $x[n]$, we can write

$$\begin{aligned} X[k] &= \sum_{n=0}^{N-1} x[n] e^{-j2\pi kn/M} && \text{for } k = 0, \dots, M-1 \\ &= \sum_{n=0}^{M-1} \bar{x}[n] e^{-j2\pi kn/M} && \text{for } k = 0, \dots, M-1 \end{aligned}$$

where $\bar{x}[n]$ is a length-M signal that is a **zero-padded** version of $x[n]$, i.e.,

$$\bar{x}[n] = \begin{cases} x[n] & n = 0, \dots, N-1 \\ 0 & \text{otherwise.} \end{cases}$$

Note that the periodic sequence $\tilde{\bar{x}}[n]$ (with period M) is not the same as the periodic sequence $\tilde{x}[n]$ (with period N). The inverse DFT of the length-M sequence $X[k]$ will return $\bar{x}[n]$, from which $x[n]$ can also be recovered.

DFT Matrix Formulation When $M > N$

As an example, consider the case with $M = 3$ and $N = 2$:

$$
\begin{aligned}
\begin{bmatrix} X[0] \\ X[1] \\ X[2] \end{bmatrix} &= \begin{bmatrix} W_3^{00} & W_3^{01} & W_3^{02} \\ W_3^{10} & W_3^{11} & W_3^{12} \\ W_3^{20} & W_3^{21} & W_3^{22} \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 1 & 1 \\ 1 & e^{-j2\pi/3} & e^{-j4\pi/3} \\ 1 & e^{-j4\pi/3} & e^{-j8\pi/3} \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \\ 0 \end{bmatrix} \\
\boldsymbol{X} &= \boldsymbol{G}\boldsymbol{x}
\end{aligned}
$$

The last $M - N$ columns of $\boldsymbol{G}$ are not used.

Case 2: $M < N$

When $M < N$, we are sampling the DTFT on a coarser grid than the usual case when $M = N$. For the length-N sequence $x[n]$, we can write

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j2\pi kn/M} \qquad \text{for } k = 0, \dots, M-1.$$

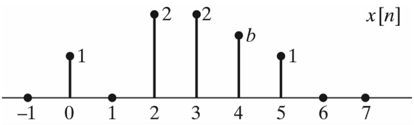
As an example, when $M = 2$ and $N = 3$, we have the matrix formulation

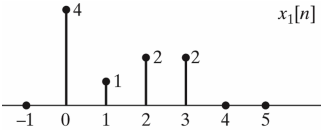
$$\begin{aligned}
\begin{bmatrix} X[0] \\ X[1] \end{bmatrix} &= \begin{bmatrix} W_2^{00} & W_2^{01} & W_2^{02} \\ W_2^{10} & W_2^{11} & W_2^{12} \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \\ x[2] \end{bmatrix} \\
&= \begin{bmatrix} 1 & 1 & 1 \\ 1 & e^{-j2\pi/2} & e^{-j4\pi/2} \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \\ x[2] \end{bmatrix} \\
\boldsymbol{X} &= \boldsymbol{G}\boldsymbol{x}
\end{aligned}$$

In general, it is not possible to uniquely recover the original length-N sequence $x[n]$ from a M-point DFT when $M < N$ due to time-domain aliasing.

Example: $N = 6$ and $M = 4$

Consider the length-6 sequence shown as $x[n]$ below. The sequence $x_1[n]$ is obtained via $\{x_1[n]\} = \text{IDFT}_4(\text{DFT}_4(\{x[n]\}))$.





The problem is to try to determine the value of b from what we know about the other values of $x[n]$ and $x_1[n]$.

Solution (part 1 of 2)

We can write out the IDFT and DFT operations with matrices ($W_N = e^{-j2\pi/N}$):

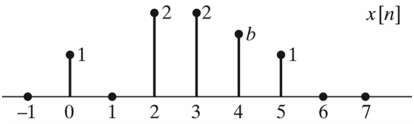
$$\begin{bmatrix} x_1[0] \\ x_1[1] \\ x_1[2] \\ x_1[3] \end{bmatrix} = \frac{1}{N} \begin{bmatrix} W_4^{00} & W_4^{01} & W_4^{02} & W_4^{03} \\ W_4^{10} & W_4^{11} & W_4^{12} & W_4^{13} \\ W_4^{20} & W_4^{21} & W_4^{22} & W_4^{23} \\ W_4^{30} & W_4^{31} & W_4^{32} & W_4^{33} \end{bmatrix}^H \times$$

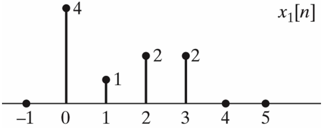
$$\begin{bmatrix} W_4^{00} & W_4^{01} & W_4^{02} & W_4^{03} & W_4^{04} & W_4^{05} \\ W_4^{10} & W_4^{11} & W_4^{12} & W_4^{13} & W_4^{14} & W_4^{15} \\ W_4^{20} & W_4^{21} & W_4^{22} & W_4^{23} & W_4^{24} & W_4^{25} \\ W_4^{30} & W_4^{31} & W_4^{32} & W_4^{33} & W_4^{34} & W_4^{35} \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ x[3] \\ x[4] \\ x[5] \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ x[3] \\ x[4] \\ x[5] \end{bmatrix}$$

Solution (part 2 of 2)

Recall our signals





and the relation

$$\begin{bmatrix} x_1[0] \\ x_1[1] \\ x_1[2] \\ x_1[3] \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ x[3] \\ x[4] \\ x[5] \end{bmatrix}$$

Since $x_1[0] = x[0] + x[4] \Rightarrow 4 = 1 + b$ we can conclude $b = 3$. Note all of the other elements in $x[n]$ and $x_1[n]$ are also consistent with this relation. This is a direct example of time-domain aliasing that occurs when $M < N$.