Digital Signal Processing
Linear Convolution with the Discrete Fourier Transform

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Linear and Circular Convolution Properties

Recall the (linear) convolution property

$$x_3[n] = x_1[n] * x_2[n] \qquad \leftrightarrow \qquad X_3(e^{j\omega}) = X_1(e^{j\omega})X_2(e^{j\omega}) \quad \forall \omega \in \mathbb{R}$$

if the necessary DTFTs exist. If $x_1[n]$ is length N_1 and $x_2[n]$ is length N_2, then $x_3[n]$ will be length $N_3 = N_1 + N_2 - 1$. See MATLAB function `conv`.

For the DFT, we have the circular convolution property

$$x_3[n] = x_1[n] \, \textcircled{N} \, x_2[n] \quad \leftrightarrow \quad X_3[k] = X_1[k]X_2[k] \quad \forall k = 0, \dots, N-1$$

where

$$x_1[n] \, \textcircled{N} \, x_2[n] = \sum_{k=0}^{N-1} x_1[k] x_2[((n-k))_N] = \sum_{k=0}^{N-1} x_1[((n-k))_N] x_2[k].$$

Note that $x_1[n]$ and $x_2[n]$ must have the same length $N = N_1 = N_2$ and the result $x_3[n]$ will also be length N. See Matlab function `cconv`.

Linear Convolution with the DFT?

Suppose we want to compute

$$x_3[n] = x_1[n] * x_2[n].$$

We could compute the DTFTs of $x_1[n]$ and $x_2[n]$, take their product, and then compute the inverse DTFT to get $x_3[n]$, i.e.,

$$x_3[n] = \mathsf{IDTFT}(\mathsf{DTFT}(x_1[n]) \cdot \mathsf{DTFT}(x_2[n]))$$

What if we want to use the DFT to compute the linear convolution instead? We know

$$x_3[n] = \mathsf{IDFT}(\mathsf{DFT}(x_1[n]) \cdot \mathsf{DFT}(x_2[n]))$$

will not work because this performs circular convolution.

Avoiding Time-Domain Aliasing

Recall our notation $W_M = e^{-j2\pi/M}$. We have seen previously that the M-point DFT of a finite-length sequence $x_i[n]$ with length N_i

$$X_i[k] = \sum_{n=0}^{N_i-1} x_i[n] W_M^{kn} \quad k = 0, 1, \dots, M-1$$

must satisfy $M \geq N_i$ to avoid time-domain aliasing when computing $\text{IDFT}_M(\text{DFT}_M(x_i[n]))$.

If $x_3[n] = x_1[n] * x_2[n]$ with $x_1[n]$ and $x_2[n]$ both finite length sequences, then the longest sequence is $x_3[n]$ with length $N_3 = N_1 + N_2 - 1$.

This implies that our DFTs $X_1[k]$, $X_2[k]$, and $X_3[k]$ should all be of length $M \geq N_1 + N_2 - 1$ to avoid time-domain aliasing. In other words,

$$x_3[n] = \text{IDFT}_M(\text{DFT}_M(x_1[n]) \cdot \text{DFT}_M(x_2[n]))$$

will result in $x_3[n] = x_1[n] * x_2[n]$ if $M \geq N_1 + N_2 - 1$.

Example

Suppose $x_1 = [\underline{1}, 2, 3]$ and $x_2 = [\underline{1}, 1, 1]$. We can compute the linear convolution as

$$x_3[n] = x_1[n] * x_2[n] = [\underline{1}, 3, 6, 5, 3].$$

If we instead compute

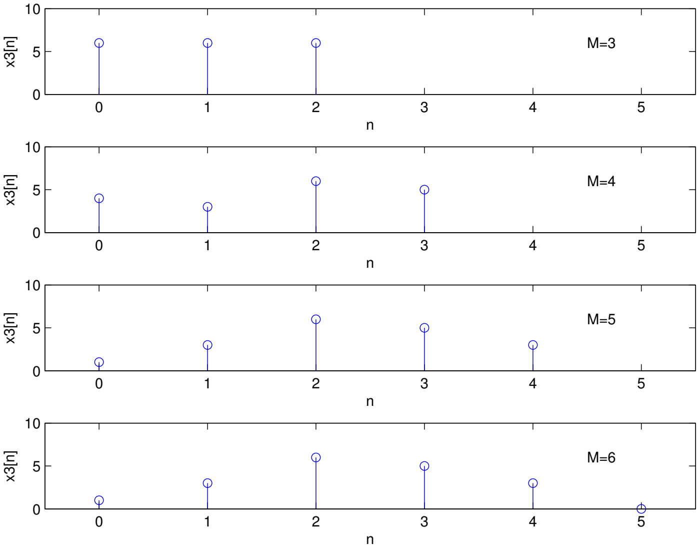
$$x_3[n] = \mathsf{IDFT}_M(\mathsf{DFT}_M(x_1[n]) \cdot \mathsf{DFT}_M(x_2[n]))$$

we get

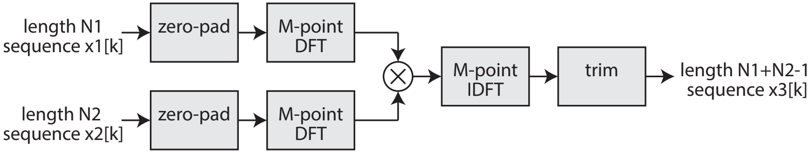
$$x_3[n] = \begin{cases} [\underline{6}, 6, 6] & M = 3 \\ [\underline{4}, 3, 6, 5] & M = 4 \\ [\underline{1}, 3, 6, 5, 3] & M = 5 \\ [\underline{1}, 3, 6, 5, 3, 0] & M = 6 \end{cases}$$

Observe that time-domain aliasing of $x_3[n]$ is avoided for $M \geq 5$.

Example (continued)



Linear Convolution with the DFT



Remarks:

- Zero-padding avoids time-domain aliasing and make the circular convolution behave like linear convolution.
- M should be selected such that $M \geq N_1 + N_2 - 1$.
- In practice, the DFTs are computed with the FFT.
- The amount of computation with this method can be less than directly performing linear convolution (especially for long sequences).
- Since the FFT is most efficient for sequences of length 2^m with integer m, M is usually chosen so that $M = 2^m \geq N_1 + N_2 - 1$.