Digital Signal Processing
Fourier Analysis of Continuous-Time Signals with the Discrete Fourier Transform

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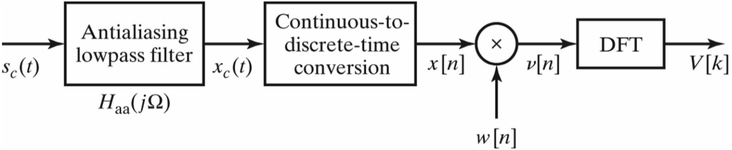
Fourier Analysis of CT Signals with the DFT

Scenario:



Ideally, would like to compute the DTFT of the sequence $x[n]$ since we have a direct relationship between $X(e^{j\omega})$ and $X_c(j\Omega)$, i.e.,

$$X(e^{j\omega}) = \frac{1}{T} \sum_{r=-\infty}^{\infty} X_c\left(j\left(\frac{\omega}{T} - \frac{2\pi r}{T}\right)\right)$$

In practice, we can usually only compute the DFT (via the FFT). Since the DFT operates on a finite number of samples, we usually must apply a window function $w[n]$ to the sequence $x[n]$ prior to computing the DFT.

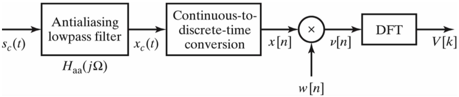
Common Window Functions (window length $M+1$)

TABLE 7.2 COMPARISON OF COMMONLY USED WINDOWS

Type of Window	Peak Side-Lobe Amplitude (Relative)	Approximate Width of Main Lobe	Peak Approximation Error, $20\log_{10}\delta$ (dB)	Equivalent Kaiser Window, β	Transition Width of Equivalent Kaiser Window
Rectangular	−13	$4\pi/(M+1)$	−21	0	$1.81\pi/M$
Bartlett	−25	$8\pi/M$	−25	1.33	$2.37\pi/M$
Hann	−31	$8\pi/M$	−44	3.86	$5.01\pi/M$
Hamming	−41	$8\pi/M$	−53	4.86	$6.27\pi/M$
Blackman	−57	$12\pi/M$	−74	7.04	$9.19\pi/M$

- Main lobe width $\Delta_{\rm ML}$ is defined as the distance between the first nulls in the DTFT of the window $W(e^{j\omega})$.
- Relative side lobe amplitude $A_{\rm SL}$ is the ratio (in dB) of the amplitude of the main lobe to the amplitude of the largest side lobe. The sign of this parameter is inconsistent.

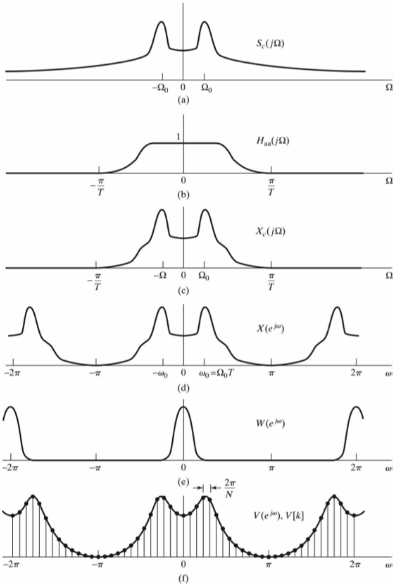
Effect of Window Function



Recall that multiplication in the time domain leads to convolution in the frequency domain, i.e.,

$$V(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\gamma})W(e^{j(\omega-\gamma)})\,d\gamma$$

One effect of the window is that it "blurs" any sharp features in the original DTFT $X(e^{j\omega})$.



Computing the DFT: Relating k and Ω

Recall that the DFT is computed on the windowed sequence $v[n]$ of length $M+1$. The length-N DFT (usually $N = M+1$) is then

$$V[k] = \sum_{n=0}^{M} v[n] W_N^{kn}$$

for $k = 0, \dots, N-1$ with $W_N = e^{-j2\pi/N}$.

Since

$$V[k] = V(e^{j\omega})|_{\omega = 2\pi k/N}$$

and $\omega = \Omega T$ we can relate the frequency index k to the original continuous time frequency as

$$\Omega = \frac{2\pi k}{TN} = \frac{\Omega_s k}{N}$$

where $\Omega_s = \frac{2\pi}{T}$ is the radian sampling frequency.

Using `fftshift`

When computing the DFT in MATLAB, we use the `fft` command. This returns $V[k]$ for $k = 0, \dots, N-1$ which corresponds to frequencies

$$\Omega = 0, \frac{\Omega_s}{N}, \frac{2\Omega_s}{N}, \dots, \frac{(N-1)\Omega_s}{N}.$$

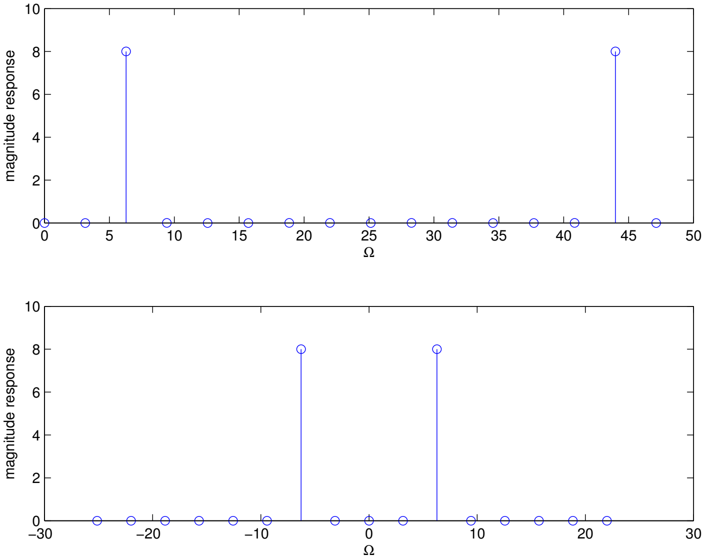
Sometimes we prefer to visualize the frequency response over $\frac{-\Omega_s}{2}$ to $\frac{\Omega_s}{2}$ We can use the MATLAB command `fftshift` to swap the left and right halves of $X[k]$ so that

$$\Omega = \frac{-\frac{N}{2}\Omega_s}{N}, \frac{\left(-\frac{N}{2}+1\right)\Omega_s}{N}, \dots, \frac{\left(\frac{N}{2}-1\right)\Omega_s}{N}$$

Example:

```
N = 16;                 % signal length
T = 1/8;                % sampling period
n = 0:N-1;              % discrete time indices
Omega_s = 2*pi/T;       % radian sampling frequency
v = sin(2*pi*n*T);      % make finite-length signal
subplot(2,1,1); stem(n*Omega_s/N,abs(fft(v)));
xlabel('\Omega'); ylabel('magnitude response');
subplot(2,1,2); stem((n-8)*Omega_s/N,abs(fftshift(fft(v))));
xlabel('\Omega'); ylabel('magnitude response');
```

`fftshift` Example



Choosing the Window and DFT Length N: Example

Suppose we have a continuous-time signal

$$s_c(t) = \sin(2\pi 970 t) + \sin(2\pi 990 t)$$

and we want the two frequencies present in this signal to line up exactly at separate DFT frequency indices k_1 and k_2 without aliasing. Assume a rectangular window.

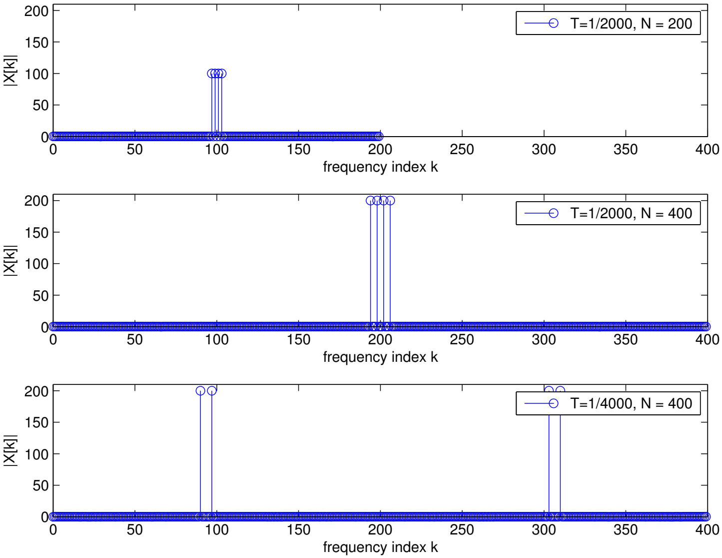
Since $\Omega = \frac{\Omega_s k}{N}$ and $\Omega_s = \frac{2\pi}{T}$, we can write

$$970 \cdot NT = k_1$$
$$990 \cdot NT = k_2$$

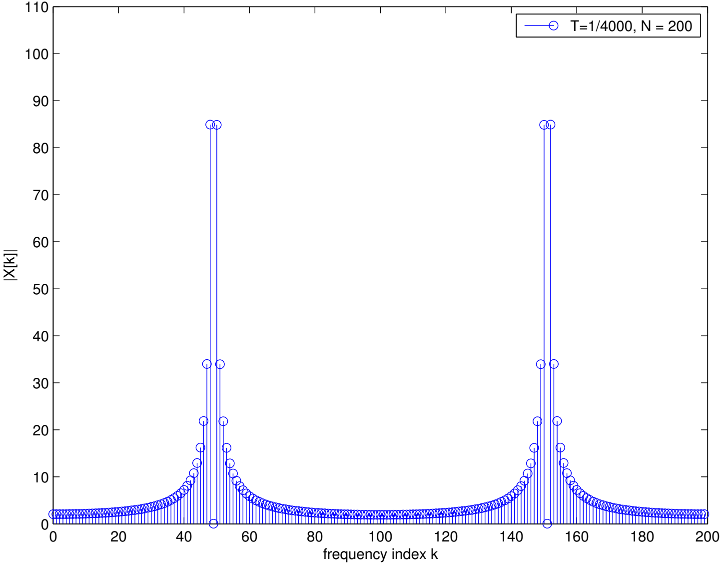
where N, k_1, and k_2 are all integers. Some possible solutions:

- $T = \frac{1}{2000}$ and $N = 200$ so that $k_1 = 97$ and $k_2 = 99$
- $T = \frac{1}{2000}$ and $N = 400$ so that $k_1 = 194$ and $k_2 = 198$
- $T = \frac{1}{4000}$ and $N = 400$ so that $k_1 = 97$ and $k_2 = 99$

Example Continued



Frequencies Corresponding to Non-Integer DFT Indices



Remarks

- In the first three examples, the signal $v[n]$ contained an integer number of periods of $s_c(t)$.
- In the last example, the signal $v[n]$ contained a non-integer number of periods of $s_c(t)$.
- Recall the relationship between the DFT and the DFS:

$$V[k] = \tilde{V}[k] \quad k = 0, \dots, N-1$$

Since $\tilde{v}[n] \leftrightarrow \tilde{V}[k]$ is the periodic extension of the finite-length signal $v[n] \leftrightarrow V[k]$, we get undesirable discontinuities in $\tilde{v}[n]$ if $v[n]$ contains a non-integer number of periods of $s_c(t)$.