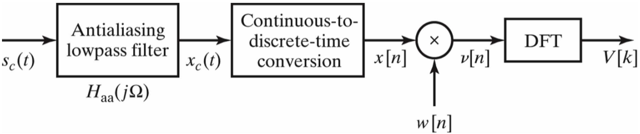
Digital Signal Processing
Discrete Fourier Transform
Analysis of Sinusoidal Signals

D. Richard Brown III

Fourier Analysis for Sinusoidal Input Signals



Suppose the input signal is a sum of weighted sinusoids such that

$$s_c(t) = \sum_i A_i \cos(\Omega_i t + \theta_i)$$

Assuming ideal sampling, we can use Euler's identity and the frequency-shifting property of the DTFT to write

$$\begin{aligned} V(e^{j\omega}) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\gamma}) W(e^{j(\omega-\gamma)})\, d\gamma \\ &= \sum_i \frac{A_i}{2} \left\{ e^{j\theta_i} W\left(e^{j(\omega-\omega_i)}\right) + e^{-j\theta_i} W\left(e^{j(\omega+\omega_i)}\right) \right\} \end{aligned}$$

For sinusoidal inputs, the DTFT of $v[n]$ consists of a sum of scaled and frequency shifted DTFTs of the window function $w[n]$.

Rectangular Window Function

The rectangular window function of length L is defined as

$$w_r[n] = \begin{cases} 1 & n = 0, \dots, L-1 \\ 0 & \text{otherwise.} \end{cases}$$

The DTFT can be computed as

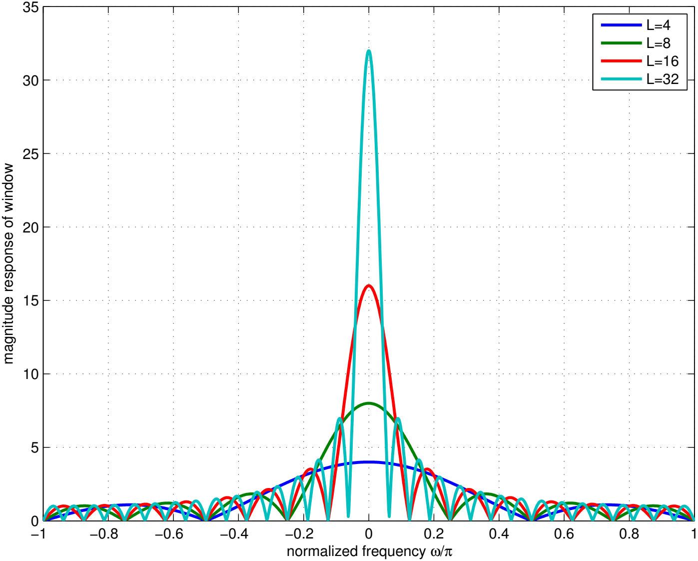
$$W_r(e^{j\omega}) = \sum_{l=0}^{L-1} e^{-j\omega n} = e^{-j\omega\left(\frac{L-1}{2}\right)} \frac{\sin(\omega L/2)}{\sin(\omega/2)}$$

Observe that $|W_r(e^{j\omega})| = 0$ when $\omega = \frac{k2\pi}{L}$ for integer $k \neq 0$. Hence, the "main lobe width" of the rectangular window is

$$\Delta_{\text{ML}} = \frac{4\pi}{L}.$$

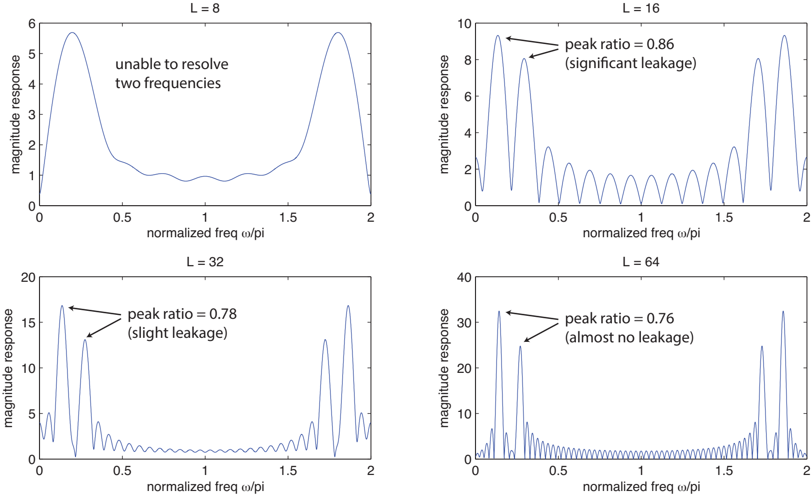
Among all windows, this is the narrowest main lobe width for a given L.

Rectangular Windows of Different Lengths



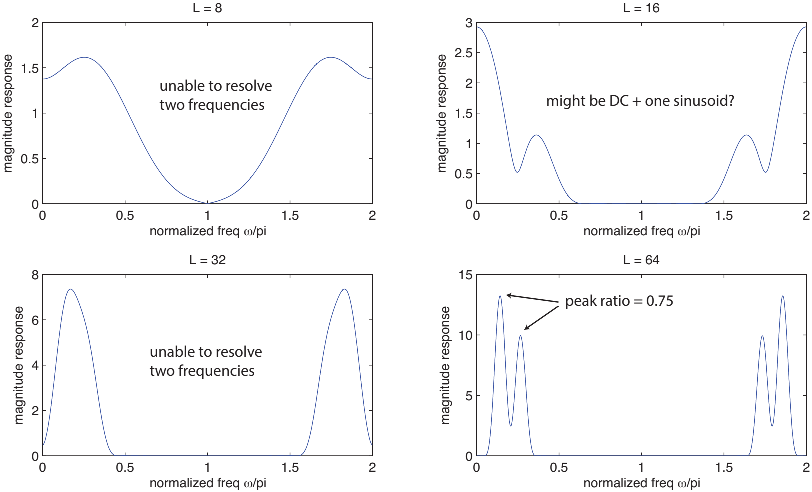
Example: Window Length and DTFT Frequency Resolution

Consider the windowed input signal $v[n] = \cos\left(\frac{2\pi}{14}n\right) + 0.75\cos\left(\frac{4\pi}{15}n\right)$ for $n = 0, \ldots, L-1$ and equal to zero otherwise.



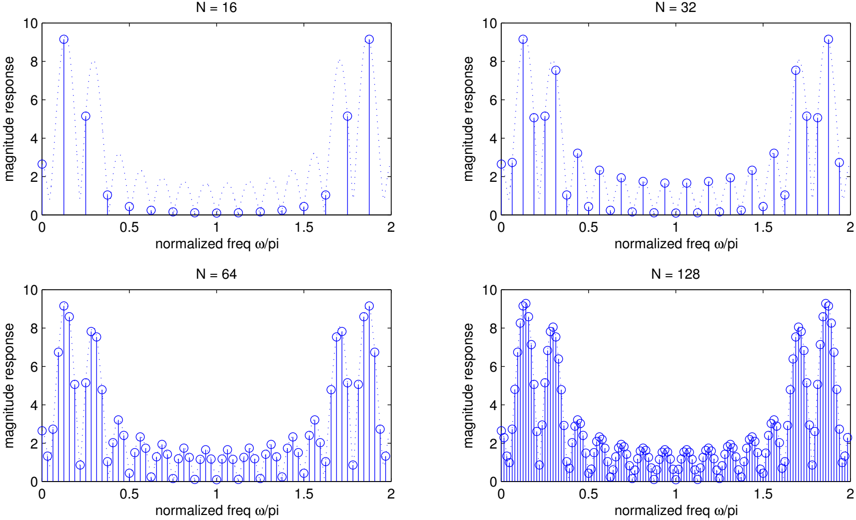
Example: Window Length and DTFT Frequency Resolution

Same example, but using a Blackman window instead of a rectangular window:



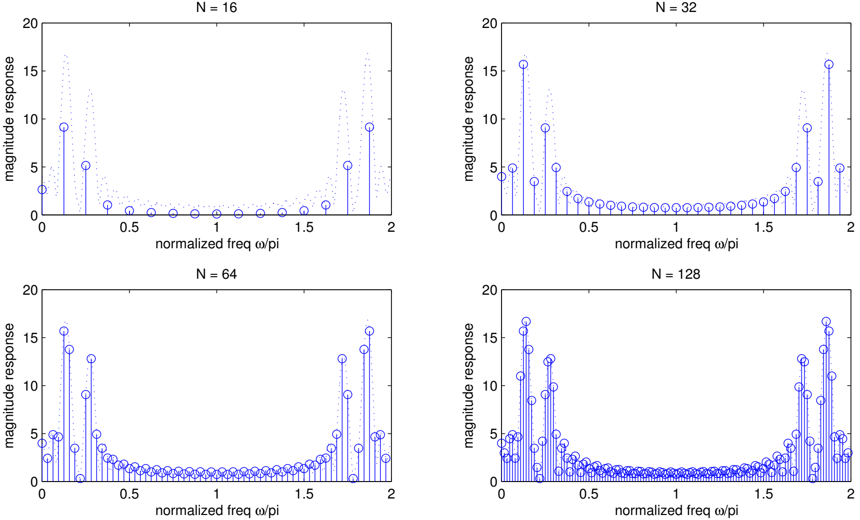
DFT vs. DTFT: Effect of Spectral Sampling ($L = 16$)

Rectangular window.



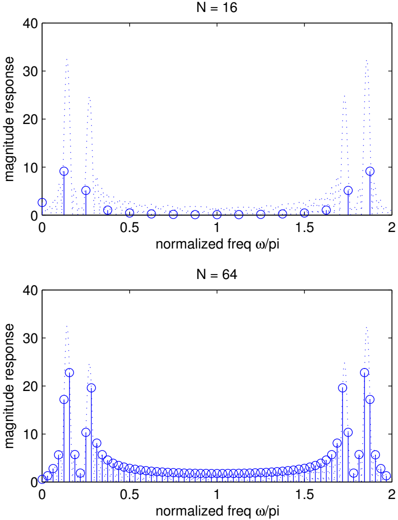
DFT vs. DTFT: Effect of Spectral Sampling ($L = 32$)

Rectangular window.



DFT vs. DTFT: Effect of Spectral Sampling ($L = 64$)

Rectangular window.



Remarks

1. Windowing can have a significant effect on the analysis of sinusoidal signals with the DFT
   - Reduced resolution (want narrow main lobe width)
   - Spectral leakage (want low side lobes)
2. All common windows have an approximate main lobe width inversely proportional to the window length L.
3. Relative side lobe amplitudes are fixed for rectangular, Bartlett, Hann, Hamming, and Blackman windows.
4. **Rectangular window**: narrowest main lobe width for a given window length L, but has the largest side lobes.
5. **Blackman window**: widest main lobe width for a given window length L, but has very low side lobes.
6. **Kaiser window**: These windows have an additional parameter β that allows for a trade off main lobe width and side lobe amplitude.