Digital Signal Processing
Introduction to the z-Transform

D. Richard Brown III

The z-Transform

Recall the DTFT

$$\text{DTFT}(\{x[n]\}) = X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} = \sum_{n=-\infty}^{\infty} x[n](e^{j\omega})^{-n}.$$

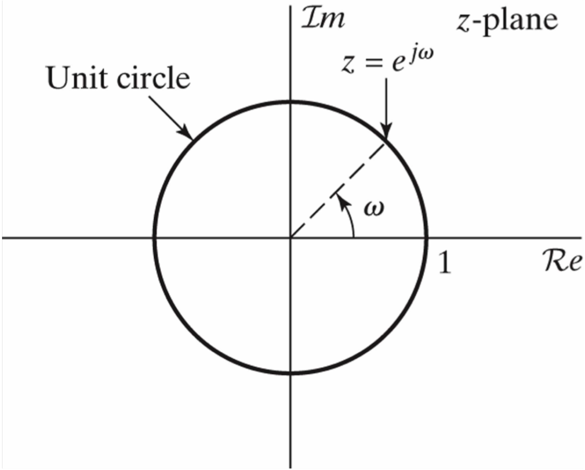
We can generalize this idea by replacing $e^{j\omega}$ with a complex variable $z = re^{j\omega}$.

Definition (bilateral z-transform):

$$\mathcal{Z}(\{x[n]\}) = X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

$X(z) : \mathbb{C} \mapsto \mathbb{C}$ should be thought of as a function of the complex variable z. The set of values $z \in \mathcal{S} \subset \mathbb{C}$ for which this sum converges is called the "region of convergence" (ROC).

The Complex Plane



In general, the z-transform is specified by both the function $X(z)$ and the ROC, e.g., $a < |z| < b$.

Relationship with the DTFT

In some cases, $X(e^{j\omega}) = X(z)|_{z=e^{j\omega}}$. For example, $x[n] = \delta[n-1]$

$$X(e^{j\omega}) = e^{-j\omega} \qquad X(z) = z^{-1}$$

Since z is a complex number, it has a magnitude and phase, i.e. $z = re^{j\omega}$. Hence, we can write

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] \underbrace{r^{-n} e^{-j\omega n}}_{z^{-n}} = \sum_{n=-\infty}^{\infty} \underbrace{x[n] r^{-n}}_{g[n]} e^{-j\omega n} = \sum_{n=-\infty}^{\infty} g[n] e^{-j\omega n}.$$

The z-transform can be thought of as the DTFT of the modified sequence $g[n] = x[n]r^{-n}$. Even in cases when the DTFT of $x[n]$ doesn't converge, the DTFT of $g[n]$ may converge for some values of r.

Convergence Example

The DTFT doesn't uniformly converge for many interesting sequences, e.g.

$$\text{DTFT}(u[n]) = \sum_{n=0}^{\infty} e^{-j\omega n} = ? \quad \text{(not absolutely summable)}$$

The z-transform uniformly converges for a broader class of sequences, e.g.

$$\begin{aligned}
\mathcal{Z}(u[n]) &= \sum_{n=0}^{\infty} z^{-n} \\
&= \lim_{N\to\infty} \left(1 + z^{-1} + \cdots + z^{-N}\right) \\
&= \lim_{N\to\infty} \frac{1 - z^{-N-1}}{1 - z^{-1}} \\
&= \frac{1}{1 - z^{-1}}
\end{aligned}$$

with ROC $|z| > 1$. Also, in this case $X(e^{j\omega}) \neq X(z)|_{z=e^{j\omega}}$.

Remarks and Motivation

The z-transform is **analytic** for all z in the ROC. Among other things, this means it is infinitely differentiable.

Some useful things about the z-transform:

- Generalization of the DTFT.
- Convergence for a broader class of sequences than the DTFT.
- Working in z-domain is often more convenient than time or frequency domain.
- We can solve for the output of certain types of systems algebraically.
- We can easily determine the stability and causality of a system.