Digital Signal Processing
z-Transform Region of Convergence

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Region of Convergence

We define the region of convergence $\mathcal{S} \subset \mathbb{C}$ of the sequence $\{x[n]\}$ as

$$\mathcal{S} = \left\{ z \in \mathbb{C} : \sum_{n=-\infty}^{\infty} |x[n]z^{-n}| < \infty \right\}$$

Remarks:

- Suppose you know the sum above converges for a particular $z_1 = r_1 e^{j\omega_1}$. Then it converges for all z with $|z| = |z_1| = r_1$.
- The ROC is important because different sequences can have the same z-transform, i.e., the z-transform is not unique without its ROC.
- When we specify the Z-transform of a sequence, we also must specify its ROC (except for certain special cases):

$$x[n] \overset{\mathcal{Z}}{\longleftrightarrow} X(z) \qquad \mathrm{ROC} : \mathcal{S}$$

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Example 1: $x[n] = u[n]$. The ROC is all $z \in \mathbb{C}$ such that $\sum_{n=0}^{\infty} |z|^{-n} < \infty$. We know this sum is finite only if $|z| > 1$. Hence the ROC of $x[n] = u[n]$ is $\mathcal{S} = \{z \in \mathbb{C} : |z| > 1\}$. For $z \in \mathcal{S}$, we have

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = \sum_{n=0}^{\infty} z^{-n} = \sum_{n=0}^{\infty} (z^{-1})^n = \frac{1}{1 - z^{-1}}.$$

Example 2: $x[n] = -u[-n-1]$. The ROC is all $z \in \mathbb{C}$ such that $\sum_{n=-\infty}^{-1} |z|^{-n} = \sum_{n=1}^{\infty} |z|^{n} < \infty$. We know this sum is finite only if $|z| < 1$. Hence the ROC of $x[n] = -u[-n-1]$ is $\mathcal{S} = \{z \in \mathbb{C} : |z| < 1\}$. For $z \in \mathcal{S}$, we have

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = -\sum_{n=-\infty}^{-1} z^{-n} = -\sum_{n=1}^{\infty} z^{n} = -\frac{z}{1-z} = \frac{1}{1 - z^{-1}}.$$

Same $X(z)$ but different ROC.

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Example 3: $x[n] = \alpha^n$ for $\alpha \in \mathbb{C}$. We can write

$$\begin{aligned}
X(z) &= \sum_{n=-\infty}^{\infty} x[n] z^{-n} \\
&= \sum_{n=-\infty}^{-1} \alpha^n z^{-n} + \sum_{n=0}^{\infty} \alpha^n z^{-n} \\
&= \sum_{n=1}^{\infty} \alpha^{-n} z^n + \sum_{n=0}^{\infty} \alpha^n z^{-n} \\
&= \sum_{n=1}^{\infty} (\alpha^{-1} z)^n + \sum_{n=0}^{\infty} (\alpha z^{-1})^n
\end{aligned}$$

The first sum is finite for what values of $z \in \mathbb{C}$?

The second sum is finite for what values of $z \in \mathbb{C}$?

What can we say about the ROC?

The z-Transform and the DTFT

Recall

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} \qquad \text{and} \qquad X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

It is clear that the DTFT is a special case of the z-transform with $z = e^{j\omega}$.

The DTFT converges if and only if the ROC of $X(z)$ includes the ring $|z| = 1$. It is incorrect to just substitute $X(e^{j\omega}) = X(z)|_{z=e^{j\omega}}$ if the ROC of $X(z)$ does not include the unit circle.

Example: We saw earlier that, given $x[n] = u[n]$, we can compute $X(z) = \frac{1}{1-z^{-1}}$. Does $X(e^{j\omega}) = \frac{1}{1-e^{-j\omega}}$?