Digital Signal Processing
Properties of the z-Transform Region of Convergence

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Rational z-Transforms: Poles and Zeros

Many sequences of interest have rational z-transforms of the form

$$X(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_M z^{-M}}{1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_N z^{-N}}$$
$$= b_0 z^{(N-M)} \frac{\prod_{m=1}^{M}(z - \xi_m)}{\prod_{n=1}^{N}(z - \lambda_n)} = b_0 z^{(N-M)} \frac{P(z)}{Q(z)}$$

where $P(z)$ and $Q(z)$ are polynomials in z.

Definition

The **zeros** of $X(z)$ are the set of values of $z \in \mathbb{C}$ such that $X(z) = 0$.

Definition

The **poles** of $X(z)$ are the set of values of $z \in \mathbb{C}$ such that $|X(z)| = \infty$.

Rational z-Transforms: Poles and Zeros

Example: $X(z) = \frac{1}{1-z^{-1}}$ with ROC $|z| > 1$.

What are the poles?

What are the zeros?

Rational z-Transforms: Poles and Zeros

For sequences with a rational z-transform, we have:

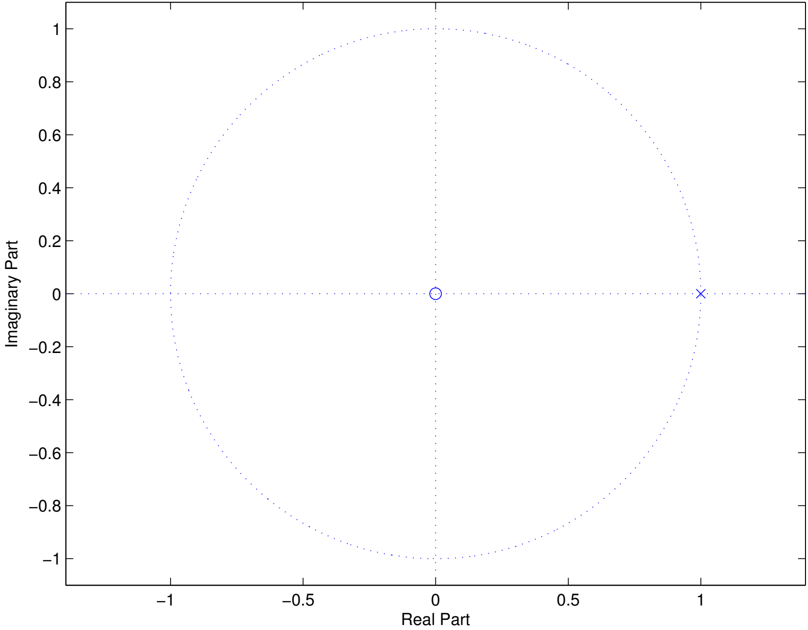
$$X(z) = b_0 z^{(N-M)} \frac{\prod_{m=1}^{M}(z-\xi_m)}{\prod_{n=1}^{N}(z-\lambda_n)} = b_0 z^{(N-M)} \frac{P(z)}{Q(z)}$$

Remarks:

- If $P(z)$ and $Q(z)$ are coprime, then the finite zeros of $X(z)$ are the roots of $P(z)$ and the finite poles of $X(z)$ are the roots of $Q(z)$.
- If $N > M$ there will be $N - M$ additional zeros at $z = 0$.
- If $N < M$, there will be $M - N$ additional poles at $z = 0$.

Matlab can easily convert from the coefficients of a rational z-transform to the pole/zeros factorization, e.g. `[z,p,k] = tf2zpk(num,den)`.

Other potentially useful Matlab functions: `roots`, `poly`, `zplane`.


1
0.8
0.6
0.4
0.2
0
–0.2
–0.4
–0.6
–0.8
–1
Imaginary Part
–1
–0.5
0
0.5
1
Real Part

ROC Properties for Rational z-Transforms (1 of 2)

1. The ROC is a ring or a disk in the z-plane centered at the origin.
2. The ROC cannot contain any poles.
3. If $\{x[n]\}$ is a **finite-length** sequence, then the ROC is the entire z-plane except possibly $z = 0$ or $|z| = \infty$.
4. If $\{x[n]\}$ is an **infinite-length right-sided** sequence, then the ROC extends outward from the largest magnitude finite pole of $X(z)$ to (and possibly including) $|z| = \infty$.
5. If $\{x[n]\}$ is an **infinite-length left-sided** sequence, then the ROC extends inward from the smallest magnitude finite pole of $X(z)$ to (and possibly including) $z = 0$.
6. If $\{x[n]\}$ is an **infinite-length two-sided** sequence, then the ROC will be a ring on the z-plane, bounded on the interior and exterior by a pole, and not containing any poles.

ROC Properties for Rational z-Transforms (2 of 2)

7. The ROC must be a connected region.
8. The DTFT of the sequence $\{x[n]\}$ converges absolutely if and only if the ROC of $X(z)$ contains the unit circle.
9. If $\{x[n]\} \overset{\mathcal{Z}}{\longleftrightarrow} X(z)$ with ROC:$\mathcal{S}_X$ and $\{y[n]\} \overset{\mathcal{Z}}{\longleftrightarrow} Y(z)$ with ROC:$\mathcal{S}_Y$, then the sequence $\{v[n]\} = \{ax[n] + by[n]\}$ will have a z-transform $\{v[n]\} \overset{\mathcal{Z}}{\longleftrightarrow} aX(z) + bY(z)$ with ROC that includes $\mathcal{S}_X \bigcap \mathcal{S}_Y$.

Note, in property 9, the ROC of $V(z) = aX(z) + bY(z)$ can be bigger than $\mathcal{S}_X \bigcap \mathcal{S}_Y$. For example:

$$x[n] = u[n] \overset{\mathcal{Z}}{\longleftrightarrow} X(z) = \frac{1}{1 - z^{-1}} \quad \text{ROC} : |z| > 1$$

$$y[n] = u[n-1] \overset{\mathcal{Z}}{\longleftrightarrow} Y(z) = \frac{z^{-1}}{1 - z^{-1}} \quad \text{ROC} : |z| > 1$$

$$v[n] = x[n] - y[n] = \delta[n] \overset{\mathcal{Z}}{\longleftrightarrow} V(z) = X(z) - Y(z) = 1 \quad \text{ROC} : \text{all } z$$