Digital Signal Processing
z-Transform Examples

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Example 1

Suppose $x[n] = \alpha^{|n|}$. Determine $X(z)$ and the ROC.

One approach: Recognize that

$$x[n] = \alpha^n u[n] + \alpha^{-n} u[-n-1] = x_1[n] + x_2[n].$$

From the z-transform table (or from direct computation of the sums), we have

$$X_1(z) = \frac{1}{1-\alpha z^{-1}} \text{ with ROC } |z| > |\alpha|$$
$$X_2(z) = \frac{-1}{1-\alpha^{-1} z^{-1}} \text{ with ROC } |z| < |\alpha^{-1}|$$

Since both $X_1(z)$ and $X_2(z)$ must converge for $X(z)$ to converge, we have the ROC for $X(z)$

$$|\alpha| < z < |\alpha^{-1}|$$

which is empty unless $|\alpha| < 1$.

Example 1 continued

So given $|\alpha| < 1$ and $|\alpha| < z < |\alpha^{-1}|$, we can compute

$$\begin{aligned} X(z) &= \sum_{n=-\infty}^{\infty} \alpha^n u[n] z^{-n} + \sum_{n=-\infty}^{\infty} \alpha^{-n} u[-n-1] z^{-n} \\ &= \frac{1}{1-\alpha z^{-1}} - \frac{1}{1-\alpha^{-1} z^{-1}} \\ &= \frac{(\alpha - \alpha^{-1}) z^{-1}}{(1-\alpha z^{-1})(1-\alpha^{-1} z^{-1})} \end{aligned}$$

Poles at $z = \alpha$ and $z = \alpha^{-1}$. Zeros at $z = 0$ and $z = \infty$.

Example 2

Suppose $X(z) = \frac{1+4z^{-1}+6z^{-2}+4z^{-3}}{3+3z^{-1}-15z^{-2}+18^{-3}-12z^{-4}}$. Then we can plot the poles and zeros as

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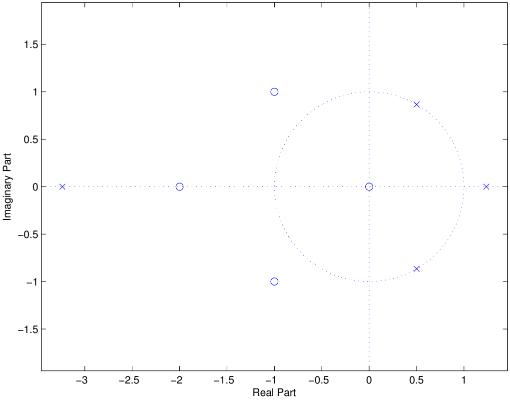
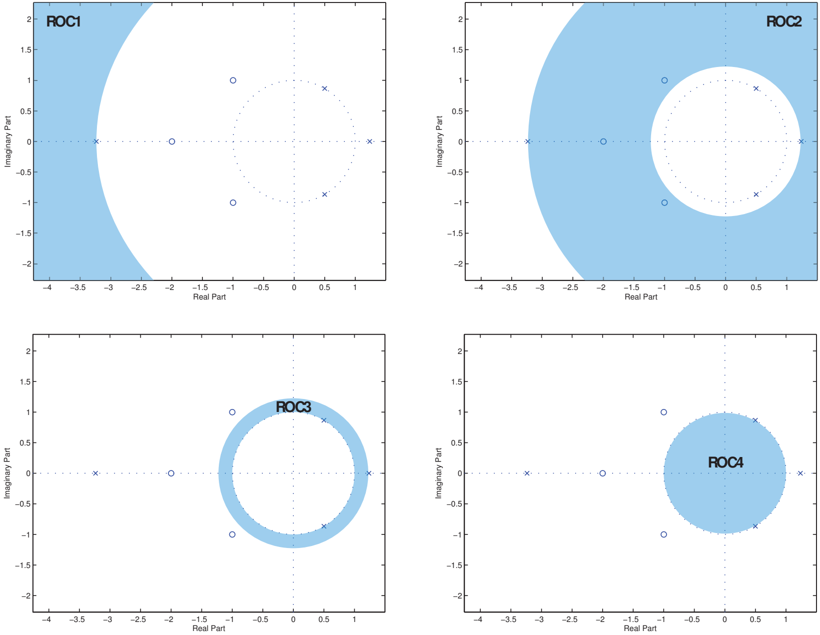
b = [1,4,6,4];
a = [3,3,-15,18,-12];
zplane(b,a);
```



Note the poles are $\lambda_1 = -3.2361$, $\lambda_2 = 1.2361$, $\lambda_3 = 0.5000 - j0.8660$, and $\lambda_4 = 0.5000 + j0.8660$. What are the possible ROCs for this $X(z)$?


ROC1
ROC2
ROC3
ROC4
Imaginary Part
Real Part
−4 −3.5 −3 −2.5 −2 −1.5 −1 −0.5 0 0.5 1
2 1.5 1 0.5 0 −0.5 −1 −1.5 −2

Example 2 continued

$$X(z) = \frac{1+4z^{-1}+6z^{-2}+4z^{-3}}{3+3z^{-1}-15z^{-2}+18^{-3}-12z^{-4}} = \sum_{i=1}^{4} \underbrace{\frac{a_i}{1-\lambda_i z^{-1}}}_{X_i(z)}$$

Note that $X_i(z)$ has an inverse z-transform that depends on the ROC

$$x_i[n] = \begin{cases} a_i \lambda_i^n u[n] & |z| > |\lambda_i| \\ -a_i \lambda_i^n u[-n-1] & |z| < |\lambda_i| \end{cases}$$

Recall that $\lambda_1 = -3.2361$, $\lambda_2 = 1.2361$, $\lambda_3 = 0.5000 - j0.8660$, and $\lambda_4 = 0.5000 + j0.8660$. Hence

ROC1: $x[n] = a_1\lambda_1^n u[n] + a_2\lambda_2^n u[n] + a_3\lambda_3^n u[n] + a_4\lambda_4^n u[n]$
ROC2: $x[n] = -a_1\lambda_1^n u[-n-1] + a_2\lambda_2^n u[n] + a_3\lambda_3^n u[n] + a_4\lambda_4^n u[n]$
ROC3: $x[n] = -a_1\lambda_1^n u[-n-1] - a_2\lambda_2^n u[-n-1] + a_3\lambda_3^n u[n] + a_4\lambda_4^n u[n]$
ROC4: $x[n] = -a_1\lambda_1^n u[-n-1] - a_2\lambda_2^n u[-n-1] - a_3\lambda_3^n u[-n-1] - a_4\lambda_4^n u[-n-1]$