Digital Signal Processing
The Inverse z-Transform

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Inverse z-Transform

The inverse z-transform is based on a special case of the Cauchy integral theorem

$$\frac{1}{2\pi j}\oint_C z^{-\ell}\,dz = \begin{cases} 1 & \ell = 1 \\ 0 & \ell \neq 1 \end{cases}$$

where C is a counterclockwise contour that encircles the origin. If we multiply $X(z)$ by z^{n-1} and compute

$$\begin{aligned}
\frac{1}{2\pi j}\oint_C X(z)z^{n-1}\,dz &= \frac{1}{2\pi j}\oint_C \sum_{m=-\infty}^{\infty} x[m]z^{-m+n-1}\,dz \\
&= \sum_{m=-\infty}^{\infty} x[m]\,\underbrace{\frac{1}{2\pi j}\oint_C z^{-(m-n+1)}\,dz}_{=1 \text{ only when } m-n+1=1} \\
&= \sum_{m=-\infty}^{\infty} x[m]\delta(m-n) \\
&= x[n]
\end{aligned}$$

Hence, the inverse z-transform of $X(z)$ is defined as $x[n] = \frac{1}{2\pi j}\oint_C X(z)z^{n-1}\,dz$ where C is a counterclockwise closed contour in the ROC of $X(z)$ encircling the origin.

Inverse z-Transform via Cauchy's Residue Theorem

Denote the unique poles of $X(z)$ as $\lambda_1, \dots, \lambda_R$ and their algebraic multiplicities as $m_1, \dots, m_R$. As long as R is finite (which is the case if $X(z)$ is rational) we can evaluate the inverse z-transform via Cauchy's residue theorem which states

$$x[n] = \frac{1}{2\pi j} \oint_C X(z) z^{n-1}\, dz = \sum_{\lambda_k \text{ inside } C} \mathrm{Res}(X(z)z^{n-1}, \lambda_k, m_k)$$

where $\mathrm{Res}(F(z), \lambda_k, m_k)$ is the "residue" of $F(z) = X(z)z^{n-1}$ at the pole λ_k with algebraic multiplicity m_k, defined as

$$\mathrm{Res}(F(z), \lambda_k, m_k) = \frac{1}{(m_k - 1)!} \left[\frac{d^{m_k-1}}{dz^{m_k-1}} \left\{ (z - \lambda_k)^{m_k} F(z) \right\} \right]_{z=\lambda_k}$$

In other words, Cauchy's residue theorem allows us to compute the contour integral by computing derivatives.

Other Methods for Computing Inverse z-Transforms

Cauchy's residue theorem works, but it can be tedious and there are lots of ways to make mistakes. The Matlab function `residuez` (discrete-time residue calculator) can be useful to check your results.

Other (typically easier) options for computing inverse z-transforms:

1. Inspection (table lookup).
2. Partial fraction expansion (only for rational z-transforms).
3. Power series expansion (can be used for non-rational z-transforms).

Inspection Method

TABLE 3.1 SOME COMMON z-TRANSFORM PAIRS

Sequence	Transform	ROC
1. $\delta[n]$	1	All z
2. $u[n]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
3. $-u[-n-1]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert < 1$
4. $\delta[n-m]$	z^{-m}	All z except 0 (if $m > 0$) or ∞ (if $m < 0$)
5. $a^n u[n]$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
6. $-a^n u[-n-1]$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert < \lvert a\rvert$
7. $na^n u[n]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert > \lvert a\rvert$
8. $-na^n u[-n-1]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert < \lvert a\rvert$
9. $\cos(\omega_0 n)u[n]$	$\frac{1-\cos(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
10. $\sin(\omega_0 n)u[n]$	$\frac{\sin(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
11. $r^n\cos(\omega_0 n)u[n]$	$\frac{1-r\cos(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$
12. $r^n\sin(\omega_0 n)u[n]$	$\frac{r\sin(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$
13. $\begin{cases} a^n, & 0 \le n \le N-1, \\ 0, & \text{otherwise} \end{cases}$	$\frac{1-a^Nz^{-N}}{1-az^{-1}}$	$\lvert z\rvert > 0$

Partial Fraction Expansion Method

If

$$X(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_M z^{-M}}{1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_N z^{-N}}$$

with all N poles distinct ("first order") then it is possible to express

$$X(z) = \underbrace{\sum_{r=0}^{M-N} B_r z^{-1}}_{\text{only if } M \geq N} + \sum_{k=1}^{N} \frac{A_k}{1 - \lambda_k z^{-1}}$$

The inverse z-transform then follows directly from linearity and table lookup (pay attention to the ROC). There are many ways to determine $A_1, \dots, A_k$, for example

$$A_k = \left[(1 - \lambda_k z^{-1}) X(z)\right]_{z=\lambda_k}$$

Power Series Expansion Method

The idea here is to write $X(z)$ as

$$X(z) = \cdots + c_{-2}z^2 + c_{-1}z + c_0 + c_1 z^{-1} + c_2 z^{-2} + \dots$$

and recognize that $x[n] = c_n$ by the definition of the z-transform. This can work even for non-rational $X(z)$.

For example, suppose $a > 0$ and

$$X(z) = e^{az^{-1}}$$

and note the ROC is the whole complex plane except $z = 0$. We can use the series expansion

$$X(z) = \sum_{k=0}^{\infty} \frac{(az^{-1})^n}{n!} = 1 + \frac{a}{1}z^{-1} + \frac{a^2}{2}z^{-2} + \frac{a^4}{6}z^{-3} + \dots$$

hence $x[n] = \frac{a^n}{n!}u[n]$.