Digital Signal Processing
Inverse z-Transform Examples

D. Richard Brown III

Inverse z-Transform via Cauchy's Residue Theorem

Suppose $X(z) = \frac{1}{1-az^{-1}}$ with ROC $|z| > |a|$.

What are the poles of $X(z)$? $\lambda_1 = a$ and $m_1 = 1$.

Now what are the poles of $X(z)z^{n-1}$?

- For $n = 0$, $X(z)z^{n-1} = \frac{z^{-1}}{1-az^{-1}} = \frac{1}{z-a}$. One pole at $z = a$.
- For $n = 1, 2, \dots$, $X(z)z^{n-1} = \frac{z^{n-1}}{1-az^{-1}} = \frac{z^n}{z-a}$. Still one pole at $z = a$.

So, for $n = 0, 1, \dots$, we can write

$$
\begin{aligned}
x[n] &= \frac{1}{2\pi j}\oint_C X(z)z^{n-1}\,dz = \frac{1}{2\pi j}\oint_C \frac{z^{n-1}}{1-az^{-1}}\,dz \\
&= \frac{1}{0!}\left[\frac{d^0}{dz^0}\left\{(z-a)\frac{z^{n-1}}{1-az^{-1}}\right\}\right]_{z=a} = \left[z^n\right]_{z=a} = a^n
\end{aligned}
$$

Continued...

Inverse z-Transform via Cauchy's Residue Theorem

When $n = -1, -2, \dots$, $X(z)z^{n-1} = \frac{z^{n-1}}{1-az^{-1}} = \frac{1}{z^{-n}(z-a)}$. We have one pole at $z = a$ and now also $-n$ poles at $z = 0$. We can write

$$\begin{aligned} x[n] &= \frac{1}{2\pi j}\oint_C X(z)z^{n-1}\,dz \\ &= \frac{1}{2\pi j}\oint_C \frac{z^{n-1}}{1-az^{-1}}\,dz \\ &= \left[\frac{d^0}{dz^0}\left\{(z-a)\frac{z^{n-1}}{1-az^{-1}}\right\}\right]_{z=a} + \frac{1}{(-n-1)!}\left[\frac{d^{-n-1}}{dz^{-n-1}}\left\{(z-0)^{-n}\frac{z^{n-1}}{1-az^{-1}}\right\}\right]_{z=0} \\ &= a^n + \frac{1}{(-n-1)!}\left[\frac{d^{-n-1}}{dz^{-n-1}}\left\{\frac{1}{z-a}\right\}\right]_{z=0} \end{aligned}$$

- For $n = -1$, the second residue is simply $\frac{1}{0!}(1/(0-a)) = -a^{-1}$.
- For $n = -2$, the second residue is $\frac{1}{1!}\left[\frac{d}{dz}\left\{\frac{1}{z-a}\right\}\right]_{z=0} = -(z-a)^{-2}|_{z=0} = -a^{-2}$.
- For $n = -3$, the second residue is $\frac{1}{2!}\left[\frac{d^2}{dz^2}\left\{\frac{1}{z-a}\right\}\right]_{z=0} = (z-a)^{-3}|_{z=0} = -a^{-3}$.
- For general $n < 0$, the second residue can be computed as $-a^n$.

Hence $x[n] = 0$ for all $n < 0$.

Inverse z-Transform via Power Series Expansion

Consider $X(z) = \frac{1}{1-z^{-2}}$ with ROC $|z| > 1$.

We can do long division to determine

$$X(z) = 1 + z^{-2} + z^{-4} + z^{-6} + \dots$$

Hence

$$x[n] = \begin{cases} 1 & n \geq 0 \text{ even} \\ 0 & \text{otherwise.} \end{cases}$$

Or, some other ways to write this are

$$x[n] = \frac{1}{2}(1 + \cos(\pi n))u[n] = \frac{1}{2}(1 + (-1)^n)u[n].$$

Inverse z-Transform via Partial Fraction Expansion

Consider again $X(z) = \frac{1}{1-z^{-2}}$ with ROC $|z| > 1$. Since the poles at ± 1 are distinct and $X(z)$ is rational and proper, we can write

$$X(z) = \frac{A_1}{1 - z^{-1}} + \frac{A_2}{1 + z^{-1}}$$

We can compute

$$A_1 = \left[(1 - z^{-1})X(z)\right]_{z=1} = \left[\frac{1}{1 + z^{-1}}\right]_{z=1} = \frac{1}{2}$$

and

$$A_2 = \left[(1 + z^{-1})X(z)\right]_{z=-1} = \left[\frac{1}{1 - z^{-1}}\right]_{z=-1} = \frac{1}{2}$$

hence from the transform table we have

$$x[n] = \frac{1}{2}(u[n] + (-1)^n u[n])$$

which is the same as our previous result.

Inverse z-Transform via Partial Fraction Expansion

Let's try $X(z) = \frac{z^{-1}}{1-2z^{-1}+z^{-2}} = \frac{z^{-1}}{(1-z^{-1})^2}$ with ROC $|z| > 1$. The repeated pole makes this a bit more difficult, but we can write

$$X(z) = \frac{C_1}{1-z^{-1}} + \frac{C_2}{(1-z^{-1})^2}.$$

We can calculate

$$C_1 = -\left\{ \frac{d}{dw} \left[(1-w)^2 X(w^{-1}) \right] \right\}_{w=1} = -\left\{ \frac{d}{dw} w \right\}_{w=1} = -1$$

and

$$C_2 = \left\{ \left[(1-w)^2 X(w^{-1}) \right] \right\}_{w=1} = -\{w\}_{w=1} = 1$$

Recalling the property that multiplication in the z-domain corresponds to convolution in the time domain, we can let $v[n] = u[n] * u[n] = \{\dots, 0, 0, \underline{1}, 2, 3, \dots\} = (n+1)u[n]$ and write

$$x[n] = -u[n] + v[n] = \{\dots, 0, 0, \underline{0}, 1, 2, \dots\} = nu[n].$$

We also could have found this via table lookup (or long division).