Digital Signal Processing
Inverse $z$-Transform Examples

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Inverse $z$-Transform via Cauchy’s Residue Theorem

Suppose $X(z) = \frac{1}{1-az^{-1}}$ with ROC $|z| > |a|$. What are the poles of $X(z)$? $\lambda_1 = a$ and $m_1 = 1$. 

Now what are the poles of $X(z)z^{n-1}$?

- For $n = 0$, $X(z)z^{n-1} = \frac{z^{-1}}{1-az^{-1}} = \frac{1}{z-a}$. One pole at $z = a$.
- For $n = 1, 2, \ldots$, $X(z)z^{n-1} = \frac{z^{n-1}}{1-az^{-1}} = \frac{z^n}{z-a}$. Still one pole at $z = a$.

So, for $n = 0, 1, \ldots$, we can write

$$x[n] = \frac{1}{2\pi j} \oint_C X(z)z^{n-1} \, dz = \frac{1}{2\pi j} \oint_C \frac{z^{n-1}}{1-az^{-1}} \, dz$$

$$= \frac{1}{0!} \left[ \frac{d^0}{dz^0} \left\{ (z - a) \frac{z^{n-1}}{1-az^{-1}} \right\} \right]_{z=a} = [z^n]_{z=a} = a^n$$

Continued…
Inverse $z$-Transform via Cauchy’s Residue Theorem

When $n = -1, -2, \ldots$, $X(z)z^{n-1} = \frac{z^{n-1}}{1-az^{-1}} = \frac{1}{z^{-n}(z-a)}$. We have one pole at $z = a$ and now also $-n$ poles at $z = 0$. We can write

$$x[n] = \frac{1}{2\pi j} \int_C X(z)z^{n-1} \, dz$$

$$= \frac{1}{2\pi j} \int_C \frac{z^{n-1}}{1-az^{-1}} \, dz$$

$$= \left[ \frac{\,^0d}{dz^0} \left\{ (z-a) \frac{z^{n-1}}{1-az^{-1}} \right\} \right]_{z=a} + \frac{1}{(-n-1)!} \left[ \frac{\,^{-n-1}d}{dz^{-n-1}} \left\{ (z-0)^{-n} \frac{z^{n-1}}{1-az^{-1}} \right\} \right]_{z=0}$$

$$= a^n + \frac{1}{(-n-1)!} \left[ \frac{\,^{-n-1}d}{dz^{-n-1}} \left\{ \frac{1}{z-a} \right\} \right]_{z=0}$$

- For $n = -1$, the second residue is simply $\frac{1}{0!}(1/(0-a)) = -a^{-1}$.
- For $n = -2$, the second residue is $\frac{1}{1!} \left[ \frac{\,^{-1}d}{dz^{-1}} \left\{ \frac{1}{z-a} \right\} \right]_{z=0} = -(z-a)^{-2}|_{z=0} = -a^{-2}$.
- For $n = -3$, the second residue is $\frac{1}{2!} \left[ \frac{\,^{-2}d}{dz^{-2}} \left\{ \frac{1}{z-a} \right\} \right]_{z=0} = (z-a)^{-3}|_{z=0} = -a^{-3}$.
- For general $n < 0$, the second residue can be computed as $-a^n$.

Hence $x[n] = 0$ for all $n < 0$. 
Consider \( X(z) = \frac{1}{1-z^{-2}} \) with ROC \(|z| > 1\).

We can do long division to determine

\[
X(z) = 1 + z^{-2} + z^{-4} + z^{-6} + \ldots
\]

Hence

\[
x[n] = \begin{cases} 
1 & n \geq 0 \text{ even} \\
0 & \text{otherwise}.
\end{cases}
\]

Or, some other ways to write this are

\[
x[n] = \frac{1}{2}(1 + \cos(\pi n))u[n] = \frac{1}{2}(1 + (-1)^n)u[n].
\]
Inverse $z$-Transform via Partial Fraction Expansion

Consider again $X(z) = \frac{1}{1 - z^{-2}}$ with ROC $|z| > 1$. Since the poles at $\pm 1$ are distinct and $X(z)$ is rational and proper, we can write

$$X(z) = \frac{A_1}{1 - z^{-1}} + \frac{A_2}{1 + z^{-1}}$$

We can compute

$$A_1 = \left[ (1 - z^{-1})X(z) \right]_{z=1} = \left[ \frac{1}{1 + z^{-1}} \right]_{z=1} = \frac{1}{2}$$

and

$$A_2 = \left[ (1 + z^{-1})X(z) \right]_{z=-1} = \left[ \frac{1}{1 - z^{-1}} \right]_{z=-1} = \frac{1}{2}$$

hence from the transform table we have

$$x[n] = \frac{1}{2} (u[n] + (-1)^n u[n])$$

which is the same as our previous result.
Inverse $z$-Transform via Partial Fraction Expansion

Let’s try $X(z) = \frac{z^{-1}}{1-2z^{-1}+z^{-2}} = \frac{z^{-1}}{(1-z^{-1})^2}$ with ROC $|z| > 1$. The repeated pole makes this a bit more difficult, but we can write

$$X(z) = \frac{C_1}{1-z^{-1}} + \frac{C_2}{(1-z^{-1})^2}.$$  

We can calculate

$$C_1 = - \left\{ \frac{d}{dw} \left[ (1-w)^2 X(w^{-1}) \right] \right\}_{w=1} = - \left\{ \frac{d}{dw} w \right\}_{w=1} = -1$$

and

$$C_2 = \left\{ (1-w)^2 X(w^{-1}) \right\}_{w=1} = - \left\{ w \right\}_{w=1} = 1$$

Recalling the property that multiplication in the $z$-domain corresponds to convolution in the time domain, we can let

$v[n] = u[n] * u[n] = \{\ldots, 0, 0, 1, 2, 3, \ldots\} = (n + 1)u[n]$ and write

$$x[n] = -u[n] + v[n] = \{\ldots, 0, 0, 0, 1, 2, \ldots\} = nu[n].$$

We also could have found this via table lookup (or long division).