Digital Signal Processing
z-Transform Properties

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TABLE 3.2 SOME z-TRANSFORM PROPERTIES

Property Number	Section Reference	Sequence	Transform	ROC
		$x[n]$	$X(z)$	R_x
		$x_1[n]$	$X_1(z)$	R_{x_1}
		$x_2[n]$	$X_2(z)$	R_{x_2}
1	3.4.1	$ax_1[n] + bx_2[n]$	$aX_1(z) + bX_2(z)$	Contains $R_{x_1} \cap R_{x_2}$
2	3.4.2	$x[n - n_0]$	$z^{-n_0} X(z)$	R_x, except for the possible addition or deletion of the origin or ∞
3	3.4.3	$z_0^n x[n]$	$X(z/z_0)$	$\lvert z_0 \rvert R_x$
4	3.4.4	$nx[n]$	$-z \dfrac{dX(z)}{dz}$	R_x
5	3.4.5	$x^*[n]$	$X^*(z^*)$	R_x
6		$\mathcal{Re}\{x[n]\}$	$\frac{1}{2}[X(z) + X^*(z^*)]$	Contains R_x
7		$\mathcal{Im}\{x[n]\}$	$\frac{1}{2j}[X(z) - X^*(z^*)]$	Contains R_x
8	3.4.6	$x^*[-n]$	$X^*(1/z^*)$	$1/R_x$
9	3.4.7	$x_1[n] * x_2[n]$	$X_1(z)X_2(z)$	Contains $R_{x_1} \cap R_{x_2}$

Linearity and Time-Shifting Example

As an example, suppose

$$x[n] = (a^n + 1)u[n-2]$$

One way to compute the z-transform in this case is to rewrite

$$x[n] = \{0, \dots, \underline{0}, 0, a^2+1, a^3+1, \dots\} = a^2 a^{n-2} u[n-2] + u[n-2]$$

Recognize that a^2 is just a constant, so we have

$$\begin{aligned} X(z) &= a^2 z^{-2} \frac{1}{1-az^{-1}} + z^{-2} \frac{1}{1-z^{-1}} \\ &= \frac{z^{-2}(a^2 - a^2 z^{-1} + 1 - az^{-1})}{(1-az^{-1})(1-z^{-1})} \end{aligned}$$

with ROC $|z| > \max\{1, |a|\}$ since this is a right-sided sequence.

Upsampling Property

Suppose L is a positive integer and

$$\tilde{x}[n] = \begin{cases} x[n/L] & n/L \text{ is an integer} \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\tilde{X}(z) = X(z^L).$$

We can see this from the definition

$$\begin{aligned} \tilde{X}(z) &= \sum_{n=\dots,-L,0,L,\dots} x[n/L] z^{-n} \\ &= \sum_{m=\dots,-1,0,1,\dots} x[m] z^{-mL} \\ &= \sum_{m=\dots,-1,0,1,\dots} x[m] (z^L)^{-m} \\ &= X(z^L) \end{aligned}$$

If the ROC for $X(z)$ is $|a| < |z| < |b|$, the ROC for $\tilde{X}(z)$ is $|a|^{1/L} < |z| < |b|^{1/L}$.

Downsampling Property (1 of 2)

Suppose L is a positive integer and

$$\tilde{x}[n] = x[nL]$$

Then

$$\tilde{X}(z) = \frac{1}{L}\sum_{r=0}^{L-1} X\left(z^{1/L}e^{-j2\pi r/L}\right)$$

To see this, compute

$$\begin{aligned}\tilde{X}(z) &= \sum_{n=-\infty}^{\infty} x[nL]z^{-n}\\ &= \sum_{m=-\infty}^{\infty} x[m]z^{-m/L}\mathbb{I}_{m=kL}\end{aligned}$$

where $\mathbb{I}_{m=kL} = 1$ when m is an integer multiple of L and zero otherwise.

Downsampling Property (2 of 2)

Note that

$$\frac{1}{L}\sum_{r=0}^{L-1} e^{j(2\pi m/L)r} = \mathbb{I}_{m=kL}$$

Hence,

$$\begin{aligned}
\tilde{X}(z) &= \sum_{m=-\infty}^{\infty} x[m] z^{-m/L} \mathbb{I}_{m=kL} \\
&= \sum_{m=-\infty}^{\infty} x[m] z^{-m/L} \frac{1}{L}\sum_{r=0}^{L-1} e^{j(2\pi m/L)r} \\
&= \frac{1}{L}\sum_{r=0}^{L-1}\sum_{m=-\infty}^{\infty} x[m] z^{-m/L} e^{j(2\pi m/L)r} \\
&= \frac{1}{L}\sum_{r=0}^{L-1}\sum_{m=-\infty}^{\infty} x[m] \left(z^{1/L} e^{-j2\pi r/L}\right)^{-m} \\
&= \frac{1}{L}\sum_{r=0}^{L-1} X\left(z^{1/L} e^{-j2\pi r/L}\right)
\end{aligned}$$

If the ROC for $X(z)$ is $|a| < |z| < |b|$ then the ROC for $\tilde{X}(z)$ is $|a|^L < |z| < |b|^L$.