Digital Signal Processing
z-Transforms and LTI Systems

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Convolution Theorem

Theorem

If $\{x[n]\} \stackrel{\mathcal{Z}}{\longleftrightarrow} X(z)$ *with ROC:*$\mathcal{S}_X$ *and* $\{y[n]\} \stackrel{\mathcal{Z}}{\longleftrightarrow} Y(z)$ *with ROC:*$\mathcal{S}_Y$*, then the sequence* $\{v[n]\} = \{x[n]\} \circledast \{y[n]\}$ *will have a* z *transform* $\{v[n]\} \stackrel{\mathcal{Z}}{\longleftrightarrow} V(z) = X(z)Y(z)$ *with ROC* $\mathcal{S}_V \supseteq \mathcal{S}_X \bigcap \mathcal{S}_Y$*.*

For an LTI system with impulse response $\{h[n]\}$, we have $y[n] = h[n] * x[n]$, hence

$$Y(z) = H(z)X(z) \text{ with ROC: } \mathcal{S}_Y$$

where $H(z)$ is the z-transform of the impulse response $\{h[n]\}$ and is commonly called the “transfer function” of the LTI system.

We often use this result to compute the output of an LTI system with a given input and impulse response without performing convolution.

Transfer Function from a Finite-Dimensional Difference Eq.

Most LTI systems of practical interest can be described by finite-dimensional constant-coefficient difference equations, e.g.

$$y[n] = \sum_{k=0}^{M-1} b_k x[n-k] - \sum_{k=1}^{N-1} a_k y[n-k]$$

Even though this system is causal, we don't require causality in the following analysis. We can take the z-transform of both sides using the time-shifting property of the z-transform to write

$$Y(z) = \sum_{k=0}^{M-1} b_k z^{-k} X(z) - \sum_{k=1}^{N-1} a_k z^{-k} Y(z)$$

and group terms to write

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M-1} b_k z^{-k}}{\sum_{k=1}^{N-1} a_k z^{-k}}.$$

From this result (and knowing the ROC), you can calculate the inverse z-transform to get the impulse response $\{h[n]\}$. This fully describes the relaxed behavior (zero state response) of the LTI system.

Transfer Function ROC

Recall the ROC properties discussed earlier. If we know certain things about the system with transfer function $H(z)$, we can apply our earlier results to specify the ROC of the transfer function as follows:

- If the transfer function only has poles at zero (corresponding to a finite-length impulse response), then its ROC is all $|z| > 0$.
- If the transfer function corresponds to a **causal** system and has poles not at zero (corresponding to an infinite-length impulse response), then the ROC extends outward from the largest magnitude finite pole of $X(z)$ to (and possibly including) $|z| = \infty$.
- If the transfer function corresponds to a **anti-causal** system and has poles not at zero (corresponding to an infinite-length impulse response), then the ROC extends inward from the smallest magnitude finite pole of $X(z)$ to (and possibly including) $z = 0$.

Transfer Function Description: Capabilities and Limitations

+ Can describe memoryless or dynamic systems.
+ Can describe causal and non-causal systems (ROC).
– Not useful for non-linear systems.
– Not useful for time-varying systems.
– No explicit access to internal behavior of system.
– Can't describe systems with non-zero initial conditions. Implicitly assumes that system is relaxed.
+ Abundance of analysis techniques. Systems are usually analyzed with **basic algebra**, not calculus.

Determining Stability from the Transfer Function

Definition

A discrete-time system is BIBO stable if, for every input satisfying

$$|x[k]| \leq M_x$$

for all $k \in \mathbb{Z}$ and some $0 \leq M_x < \infty$, the output satisfies

$$|y[k]| \leq M_y$$

for all $k \in \mathbb{Z}$ and some $0 \leq M_y < \infty$.

We know that an LTI system $\mathcal{H}$ is BIBO stable if and only if its impulse response $\{h[n]\}$ is absolutely summable, i.e.

$$\sum_{n=-\infty}^{\infty} |h[n]| < \infty.$$

This can be tricky to check. Is there an easier test using the transfer function?

Determining Stability from the Transfer Function

Observe that

$$\text{BIBO stable} \quad \Longleftrightarrow \quad \sum_{n=-\infty}^{\infty} |h[n]| < \infty \quad \Longleftrightarrow \quad \sum_{n=-\infty}^{\infty} |h[n]z^{-n}| < \infty$$

for $|z| = 1$. Hence an LTI system is BIBO stable if and only if the ROC of $H(z)$ includes the unit circle. This condition also ensures the DTFT uniformly converges.

The rule you probably learned as an undergraduate student is that "an LTI system is BIBO stable if and only if all of the poles of $H(z)$ are inside the unit circle". Does this agree with the condition above?

Example: Suppose

$$H(z) = \frac{1}{1 - 2z^{-1}} \quad \text{ROC} : |z| < 2$$

Is this system BIBO stable?