Digital Signal Processing
The Unilateral z-Transform

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Unilateral z-Transform

Definition:

$$X(z) = \sum_{n=0}^{\infty} x[n] z^{-n}$$

Remarks:

- The unilateral z-transform ignores $x[-1], x[-2], \dots$ and, hence, is typically only used for sequences that are zero for $n < 0$ (sometimes called causal sequences).
- If $x[n] = 0$ for all $n < 0$ then the unilateral and bilateral transforms are identical.
- Linear.
- No need to specify the ROC (extends outward from largest pole).
- Inverse z-transform is unique (right-sided).
- Can handle non-zero initial conditions.

Unilateral z-Transform: Time-Shifting Property (1 of 2)

Note that the time-shifting property of the unilateral z-transform is different than the time-shifting property of the bilateral z-transform. Suppose $\tilde{x}[n] = x[n-1]$. Then

$$\begin{aligned}\tilde{X}(z) &= \sum_{n=0}^{\infty} x[n-1]z^{-n} \\ &= x[-1]z^{0} + x[0]z^{1} + x[1]z^{2} + \dots \\ &\overset{m=n-1}{=} x[-1] + \sum_{m=0}^{\infty} x[m]z^{-m-1} \\ &= x[-1] + z^{-1}\sum_{m=0}^{\infty} x[m]z^{-m} \\ &= x[-1] + z^{-1}X(z).\end{aligned}$$

Unilateral z-Transform: Time-Shifting Property (2 of 2)

In general, for $\tilde{x}[n] = x[n-n_d]$ we have

$$\begin{aligned}\tilde{X}(z) &= \sum_{n=0}^{\infty} x[n-n_d]z^{-n} \\ &= \sum_{m=1}^{n_d} x[m-n_d-1]z^{-m+1} + z^{-n_d}X(z).\end{aligned}$$

Example (part 1 of 3)

Suppose we have a system given by

$$y[n] - 5y[n-1] + 6y[n-2] = 3x[n-1] + 5x[n-2]$$

with initial conditions $y[-1] = \frac{11}{6}$, $y[-2] = \frac{37}{36}$ and the input $x[n] = (0.5)^n u[n]$. Determine the output $y[n]$ for all $n \geq 0$.

Take unilateral z-transforms of both sides to get

$$Y(z) - 5\left(y[-1] + z^{-1}Y(z)\right) + 6\left(y[-2] + y[-1]z^{-1} + z^{-2}Y(z)\right) = 3\left(x[-1] + z^{-1}X(z)\right) + 5\left(x[-2] + x[-1]z^{-1} + z^{-2}X(z)\right)$$

We can substitute the initial conditions and simplify this a bit by noting that $x[n] = 0$ for all $n < 0$. Hence

$$Y(z) - 5\left(\frac{11}{6} + z^{-1}Y(z)\right) + 6\left(\frac{37}{36} + \frac{11}{6}z^{-1} + z^{-2}Y(z)\right) = 3z^{-1}X(z) + 5z^{-2}X(z)$$

continued...

Example (part 2 of 3)

Since

$$X(z) = \frac{1}{1 - 0.5z^{-1}}$$

We have

$$Y(z) - 5\left(\frac{11}{6} + z^{-1}Y(z)\right) + 6\left(\frac{37}{36} + \frac{11}{6}z^{-1} + z^{-2}Y(z)\right) = \frac{3z^{-1} + 5z^{-2}}{1 - 0.5z^{-1}}$$

or, equivalently

$$Y(z)(1 - 5z^{-1} + 6z^{-2}) - \frac{55}{6} + \frac{37}{6} + 11z^{-1} = \frac{3z^{-1} + 5z^{-2}}{1 - 0.5z^{-1}}.$$

This can be rearranged as

$$Y(z) = \underbrace{\frac{3z^{-1} + 5z^{-2}}{(1 - 0.5z^{-1})(1 - 2z^{-1})(1 - 3z^{-1})}}_{H(z)X(z) \text{ (relaxed initial conditions)}} + \underbrace{\frac{3 - 11z^{-1}}{(1 - 2z^{-1})(1 - 3z^{-1})}}_{\text{zero-input response}}$$

continued...

Example (part 3 of 3)

To determine $y[n]$ we can perform a partial fraction expansion to write

$$Y(z) = \frac{26/15}{1-0.5z^{-1}} + \frac{-7/3}{1-2z^{-1}} + \frac{18/5}{1-3z^{-1}}$$

and it follows from table lookup that

$$y[n] = \frac{26}{15}(0.5)^n u[n] + -\frac{7}{3}2^n u[n] + \frac{18}{5}3^n u[n]$$

Is this a BIBO stable system?

Clearly not since the output is growing without bound even though we have a bounded input. Also, the ROC here is $|z| > 3$, which does not contain the unit circle, hence the system is not BIBO stable.