Digital Signal Processing
Periodic Sampling

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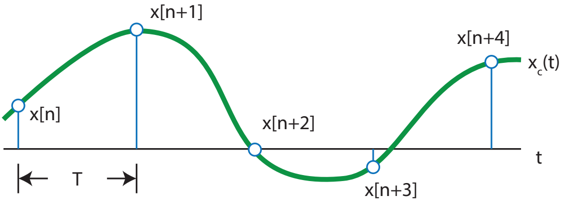
Periodic Sampling Basics

It is often the case that discrete-time signals result from periodic sampling of continuous-time signals. Given a sampling period of $T > 0$ seconds, we can say

$$x[n] = x_c(nT)$$

for $n \in \mathbb{Z}$.

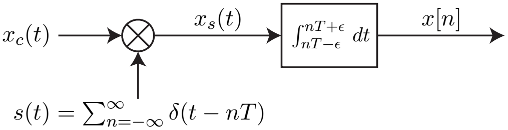


The sampling frequency is $f_s = \frac{1}{T}$ samples/sec.

For now, we focus on the ideal case where the sampling process is perfect (no quantization error, no timing jitter, etc.).

Ideal Sampling System: Time Domain Representation

To better understand sampling and reconstruction, we often represent an ideal sampler as



To see why this is an ideal sampler, we can write

$$x_s(t) = \sum_{n=-\infty}^{\infty} x_c(t)\delta(t - nT)$$

and then

$$x[n] = \int_{nT-\epsilon}^{nT+\epsilon} \sum_{m=-\infty}^{\infty} x_c(t)\delta(t - mT)\, dt = \int_{nT-\epsilon}^{nT+\epsilon} x_c(t)\delta(t - nT)\, dt = x_c(nT).$$

Poisson's Sum Formula for Continuous-Time Signals

Given a continuous-time signal $z(t)$ with CTFT $Z(\Omega)$ where the CT radian frequency is denoted as Ω, we can write the infinite sum of delayed copies of $z(t)$ as

$$\tilde{z}(t) = \sum_{n=-\infty}^{\infty} z(t - nT) = \frac{1}{T} \sum_{n=-\infty}^{\infty} Z(n\Omega_s) e^{jn\Omega_s t}$$

where $\Omega_s = 2\pi/T$.

Proof sketch: Note that $\tilde{z}(t)$ is periodic with period T and can be represented as a Fourier series

$$\tilde{z}(t) = \sum_{n=-\infty}^{\infty} \alpha_n e^{jn\Omega_s t}$$

The Fourier series coefficients can be computed as $\alpha_n = \frac{1}{T} Z(n\Omega_s)$.

Application of Poisson's Sum Formula

Recall the CTFT pair

$$\delta(t) \Longleftrightarrow 1$$

Since $s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$, we can use Poisson's sum formula to write

$$s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT) = \frac{1}{T} \sum_{n=-\infty}^{\infty} e^{jn\Omega_s t}$$

Also recall the CTFT pair

$$e^{j\Omega_0 t} \Longleftrightarrow 2\pi\delta(\Omega - \Omega_0)$$

Along with the linearity of the CTFT, this implies that

$$S(j\Omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta(\Omega - k\Omega_s).$$

Ideal Sampling System: Frequency Domain Representation

Assuming the CTFT exists for the input signal, we have

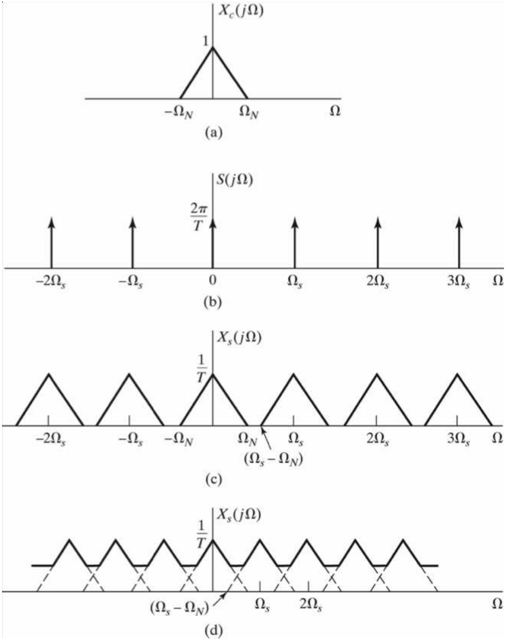
$$x_c(t) \Longleftrightarrow X_c(j\Omega)$$

$$s(t) \Longleftrightarrow S(j\Omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta(\Omega - k\Omega_s)$$

where $\Omega_s = 2\pi f_s = 2\pi/T$. Since $x_s(t) = x_c(t)s(t)$, we can use the time-domain modulation property to write

$$X_s(j\Omega) = \frac{1}{2\pi} X_c(j\Omega) * \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta(\Omega - k\Omega_s)$$

$$= \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c(j(\Omega - k\Omega_s))$$


$X_c(j\Omega)$
1
$-\Omega_N$
Ω_N
Ω
(a)
$S(j\Omega)$
$\frac{2\pi}{T}$
$-2\Omega_s$
$-\Omega_s$
0
Ω_s
$2\Omega_s$
$3\Omega_s$
Ω
(b)
$X_s(j\Omega)$
$\frac{1}{T}$
$-2\Omega_s$
$-\Omega_s$
$-\Omega_N$
Ω_N
Ω_s
$2\Omega_s$
$3\Omega_s$
Ω
$(\Omega_s - \Omega_N)$
(c)
$X_s(j\Omega)$
$\frac{1}{T}$
$(\Omega_s - \Omega_N)$
Ω_s
$2\Omega_s$
Ω
(d)