Digital Signal Processing
Frequency Domain Relations of Sampled Signals

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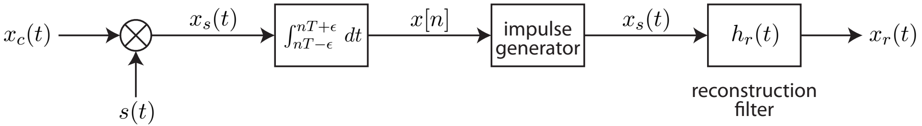
Ideal Sampling and Reconstruction

Consider



with

$$s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT).$$

We have previously related $x_s(t)$ and $x_c(t)$ as

$$x_s(t) = \sum_{n=-\infty}^{\infty} x_c(nT)\delta(t - nT)$$

and

$$X_s(j\Omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c(j(\Omega - k\Omega_s))$$

Relationship Between CTFT $X_s(j\Omega)$ and DTFT $X(e^{j\omega})$

We have

$$X_s(j\Omega) = \int_{-\infty}^{\infty} x_s(t) e^{-j\Omega t}\, dt \tag{1}$$

$$= \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} x[n]\delta(t - nT) e^{-j\Omega t}\, dt \tag{2}$$

$$= \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega nT} \tag{3}$$

Recall the DTFT of $x[n]$ is

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} \tag{4}$$

Comparison with (3) reveals that $X(e^{j\omega}) = X_s(j\Omega)|_{\Omega=\omega/T}$.

Relationship Between CTFT $X_c(j\Omega)$ and DTFT $X(e^{j\omega})$

Since $x_s(t) = x_c(t)s(t)$, we know

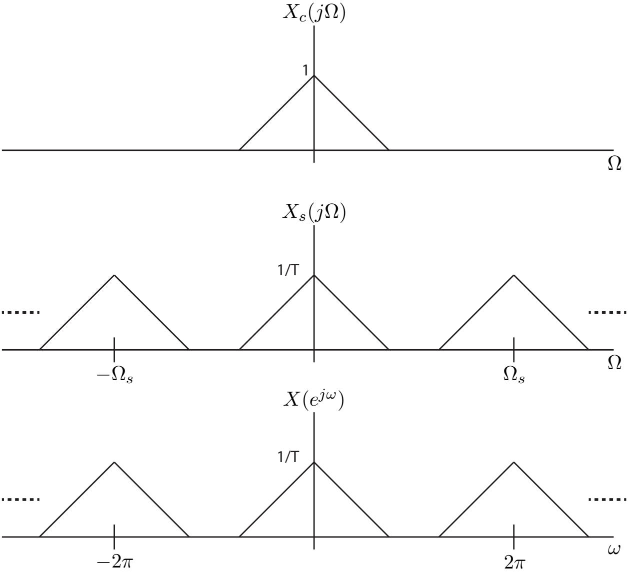
$$\begin{aligned} X_s(j\Omega) &= \frac{1}{2\pi} X_c(j\Omega) * \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta(\Omega - k\Omega_s) \\ &= \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c(j(\Omega - k\Omega_s)) \end{aligned}$$

Combining this with our previous result that $X(e^{j\omega}) = X_s(j\Omega)|_{\Omega=\omega/T}$, we get

$$\boxed{X(e^{j\omega}) = \frac{1}{T} \sum_{n=-\infty}^{\infty} X_c\left(j\left(\frac{\omega}{T} - \frac{n2\pi}{T}\right)\right)}$$

since $\Omega_s = 2\pi/T$.

Summary: No Aliasing



Summary: With Aliasing