Digital Signal Processing
Reconstruction of a Bandlimited Signal from its Samples

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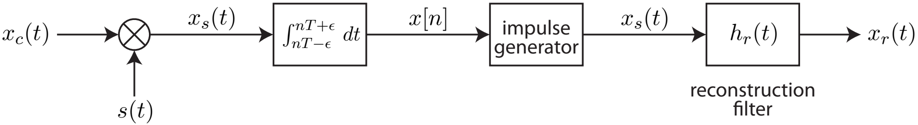
Ideal Sampling and Reconstruction

Consider



with bandlimited $x_c(t)$ such that $X_c(j\Omega) = 0$ for all $|\Omega| \geq \Omega_N$, sampling frequency $\Omega_s = \frac{2\pi}{T} \geq 2\Omega_N$, and

$$s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT).$$

Based on our previous frequency-domain analysis, we know we won't have any aliasing in this case. The ideal reconstruction filter is a lowpass filter with gain of T and any cutoff frequency Ω_c such that $\Omega_N \leq \Omega_c \leq \Omega_s - \Omega_N$.

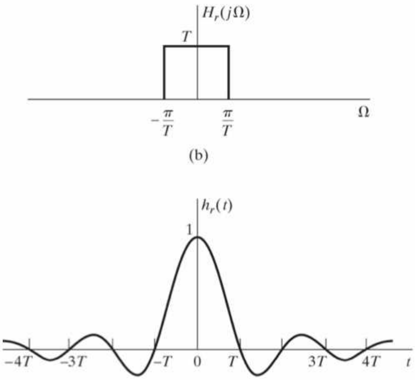
Ideal Reconstruction Filter

A convenient choice for the cutoff frequency of the reconstruction filter is $\Omega_c = \frac{\Omega_s}{2}$. This implies

$$H_r(j\Omega) = \begin{cases} T & -\frac{\pi}{T} \leq \Omega \leq \frac{\pi}{T} \\ 0 & \text{otherwise.} \end{cases}$$

We can compute the inverse CTFT to get

$$h_r(t) = \frac{\sin(\pi t/T)}{\pi t/T}$$



Note that $h_r(0) = 1$ by L'Hopital's rule and $h_r(nT) = 0$ for all $n \in \mathbb{Z} \backslash 0$.

Output of Reconstruction Filter

At the input of the reconstruction filter, we have

$$x_s(t) = \sum_{k=-\infty}^{\infty} x_c(kT)\delta(t-kT).$$

We can write the output as

$$\begin{aligned} x_r(t) &= x_s(t) * h_r(t) \\ &= \left(\sum_{k=-\infty}^{\infty} x_c(kT)\delta(t-kT)\right) * \left(\frac{\sin(\pi t/T)}{\pi t/T}\right) \\ &= \sum_{k=-\infty}^{\infty} x_c(kT)\frac{\sin(\pi(t-kT)/T)}{\pi(t-kT)/T} \end{aligned}$$

Observe that

$$x_r(nT) = \sum_{k=-\infty}^{\infty} x_c(kT)\frac{\sin(\pi(nT-kT)/T)}{\pi(nT-kT)/T} = \sum_{k=-\infty}^{\infty} x_c(kT)\delta[n] = x_c(nT).$$

Sinc Interpolation

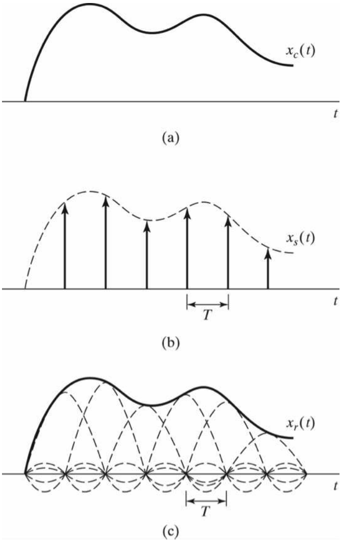
The interpolation formula

$$x_r(t) = \sum_{k=-\infty}^{\infty} x[k] \frac{\sin(\pi(t-kT)/T)}{\pi(t-kT)/T}$$

is called the Whittaker–Shannon interpolation formula and converges absolutely and locally uniformly if

$$\sum_{n \in \mathbb{Z} \setminus 0} \left| \frac{x[n]}{n} \right| < \infty.$$

This is a sufficient condition but is not necessary.



(a)
(b)
(c)

Ideal Reconstruction

To show that we can reconstruct the original signal exactly at all t from its samples $\{x[n]\}$, it is much more convenient to work in the frequency domain. Since

$$x_r(t) = \sum_{k=-\infty}^{\infty} x[k] h_r(t - kT)$$

we can write

$$\begin{aligned} X_r(j\Omega) &= \sum_{k=-\infty}^{\infty} x[k] H_r(j\Omega) e^{-j\Omega T k} \\ &= H_r(j\Omega) \sum_{k=-\infty}^{\infty} x[k] e^{-j\Omega T k} \\ &= H_r(j\Omega) X(e^{j\Omega T}) \\ &= H_r(j\Omega) \cdot \frac{1}{T} \sum_{n=-\infty}^{\infty} X_c\left(j \left(\frac{\Omega T}{T} - \frac{n2\pi}{T} \right) \right) = X_c(j\Omega) \end{aligned}$$

where the final equality uses our assumption of no aliasing.